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Mathematics

On D-equivalence Classes of some Graphs

Somayeh Jahari and Saeid Alikhani

Department of Mathematics, Yazd University, 89195-741, Yazd, Iran

(Presented by Academy Member N. Berikashvili)

ABSTRACT. Let *G* be a simple graph of order *n*. The domination polynomial of *G* is the polynomial $D(G,x)=\sum_{i=1}^{n}d(G,i)x^i$, where *d*(*G*, *i*) is the number of dominating sets of *G* of size *i*. The *n*-barbell graph Bar_n with 2*n* vertices, is formed by joining two copies of a complete graph K_n by a single edge. We prove that for every $n\geq 2$, Bar_n is not D-unique, that is, there is another non-isomorphic graph with the same domination polynomial. More precisely, we show that for every *n*, the D-equivalence class of barbell graph, [Bar_n], contains many graphs, which one of them is the complement of book graph of order *n* - 1, B_{n-1}^c. Also we present many families of graphs in D-equivalence class of $K_{n_1}\cup K_{n_2}\cup\cdots\cup K_{n_k}$. *©2016 Bull. Georg. Natl. Acad. Sci.*

Key words: domination polynomial, D-unique, equivalence, generalized barbell graphs

1. Introduction

All graphs in this paper are simple of finite orders, i.e., graphs are undirected with no loops or parallel edges and with finite number of vertices. The *complement* of a graph *G*, is a graph with the same vertex set as *G* and with the property that two vertices are adjacent in G^c if and only if they are not adjacent in *G* and is denoted by G^c. For any vertex $v\in V(G)$, the *open neighborhood* of v is the set $N(v)=\{u\in V(G)\,|\,uv\in E(G)\}$ and the *closed neighborhood* of v is the set $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V(G)$, the open neighborhood of *S* is $N(S)=\bigcup_{v\in S}N(v)$ and the closed neighborhood of *S* is $N[S]=N(S)\cup S$. A set $S\subseteq V(G)$ is a *dominating set* if $N[S]=V$, or equivalently, every vertex in $V(G)\setminus S$ is adjacent to at least one vertex in *S*. The *domination number* $\gamma(G)$, is the minimum cardinality of a dominating set in *G*. For a detailed treatment of domination theory, the reader is referred to [1]. Let $D(G,i)$ be the family of dominating sets of a graph *G* with cardinality *i* and let $d(G,i)=|D(G,i)|$.

The *domination polynomial* $D(G,x)$ of *G* is defined as $D(G,x)=\sum_{i=\gamma(G)}^{|V(G)|}d(G,i)x^i$ (see [2-4]). This polynomial is the generating polynomial for the number of dominating sets of each cardinality. Calculating the domination polynomial of a graph G is difficult in general, as the smallest power of a non-zero term is the domination number $\gamma(G)$ of the graph, and determining whether $\gamma(G) \leq k$ is known to be NP-complete [5]. But for certain classes of graphs, we can find a closed form expression for the domination polynomial. Two graphs G and H are said to be *dominating equivalent*, or simply D-equivalent, written $G \sim H$, if $D(G,x) = D(H,x)$. It is evident that the relation $\sim$ of being D-equivalence is an equivalence relation on the family Ģ of graphs, and thus Ģ is partitioned into equivalence classes, called the *D-equivalence classes*. Given $G \in$ Ģ, let

$$[G] = \{H : H \sim G\}.$$

We call $[G]$ the equivalence class determined by G. A graph G is said to be dominating unique, or simply D-unique, if $[G] = \{G\}$ [6]. Determining D-equivalence class of graphs is one of the interesting problems on equivalence classes. A question of recent interest concerning this equivalence relation $[\cdot]$ asks which graphs are determined by their domination polynomial. It is known that cycles [2] and cubic graphs of order 10 [7] (particularly, the Petersen graph) are, while if $n \equiv 0(mod 3)$, the paths of order n are not [2]. In [8], authors completely described the complete r-partite graphs which are D-unique. Their results in the bipartite case, settles in the affirmative a conjecture in [9].

Let n be any positive integer and Bar_n be n-barbell graph with $2n$ vertices which is formed by joining two copies of a complete graph K_n by a single edge. In this paper, we consider n-barbell graphs and study their domination polynomials. We prove that for every $n \geq 2$, Bar_n is not D-unique. More precisely, in Section 2, we show that for every n, $[Bar_n]$ contains many graphs, which one of them is $K_n \cup K_n$ and another one is the complement of book graph of order $n-1$, B^c_{n-1}. In Section 3, we present many graphs in $[K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}]$.

2. D-Equivalence Classes of some Graphs

In this section, we study the D-equivalence classes of some graphs. First we consider the domination polynomial of the complement of book graph. The n-book graph B_n can be constructed by bonding n copies of the cycle graph C_4 along a common edge $\{u, v\}$, see Fig. 1.

The following theorem gives a formula for the domination polynomial of B_n.

Theorem 2.1 [10] *For every* $n \in \mathbf{N}$,

$$D(B_n, x) = (x^2 + 2x)^n (2x+1) + x^2 (x+1)^{2n} - 2x^n.$$

Domination polynomials, exploring the nature and location of roots of domination polynomials of book graphs has studied in [10]. Here, we consider the domination polynomial of the complement of the book

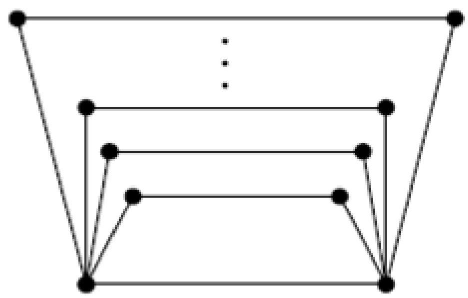

Fig. 1. The book graph B_n

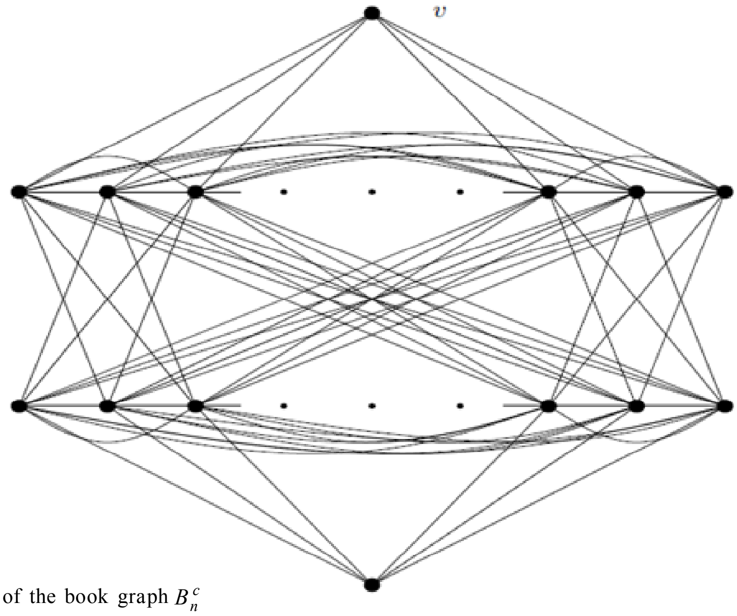

Fig. 2. Complement of the book graph B_n^c

graphs. We shall prove that the n -barbell graph Bar_n and B_{n-1}^c have the same domination polynomial.

The Turán graph $T(n,r)$ is a complete multipartite graph formed by partitioning a set of n vertices into r subsets, with sizes as equal as possible, and connecting two vertices by an edge whenever they belong to different subsets. The graph will have (n mod r) subsets of size $\lceil \frac{n}{r} \rceil$, and $r-(n \ mod \ r)$ subsets of size $\lfloor \frac{n}{r} \rfloor$. That is, a complete r-partite graph

$$K_{\lceil \frac{n}{r} \rceil, \lceil \frac{n}{r} \rceil, \ldots, \lfloor \frac{n}{r} \rfloor, \lfloor \frac{n}{r} \rfloor}.$$

The Turán graph $T(2n,n)$ can be formed by removing a perfect matching, n edges no two of which are adjacent, from a complete graph K_{2n}. As Roberts (1969) showed, this graph has boxicity exactly n; it is sometimes known as the Robert's graph [11]. If n couples go to a party, and each person shakes hands with every person except his or her partner, then this graph describes the set of handshakes that take place; for this reason it is also called the cocktail party graph. So, the cocktail party graph $CP(t)$ of order $2t$ is the graph with vertices $b_1, b_2, \cdots, b_{2t}$ in which each pair of distinct vertices form an edge with the exception of the pairs $\{b_1,b_2\},\{b_3,b_4\},\ldots,\{b_{2t-1},b_{2t}\}$. The following result is easy to obtain.

Lemma 2.2 *For every* $n \in \mathbf{N}$, $D(CP(n),x) = (1+x)^{2n} - 2nx - 1$.

Fig. 2 shows the complement of the book graph B_n^c.

The vertex contraction G/u of a graph G by a vertex u is the operation under which all vertices in $N(u)$ are joined to each other and then u is deleted (see[12]).

The following theorem is useful for finding the recurrence relations for the domination polynomials of arbitrary graphs.

Theorem 2.3 [13, 14] *Let G be a graph. For any vertex u in G we have*

$$D(G,x) = xD(G/u,x) + D(G-u,x) + xD(G-N[u],x) - (1+x)p_u(G,x),$$

where $p_u(G,x)$ is the polynomial counting the dominating sets of $G-u$ which do not contain any vertex of $N(u)$ in G.

The following theorem gives a formula for the domination polynomial of the complement of the book graph.

Theorem 2.4 *For every* $n \in \mathbf{N}$,

$$D(B_n^c,x) = ((1+x)^{n+1}-1)^2.$$

Proof. Consider graph B_n^c and vertex v in the Fig. 2. By Theorem 2.3, we have:

$$\begin{aligned} D(B_n^c,x) &= xD(B_n^c/v,x) + D(B_n^c - v,x) + xD(B_n^c - N[v],x) - (1+x)p_v(B_n^c,x) \\ &= (x+1)D(B_n^c - v,x) + xD(K_{n+1},x) - (1+x)(D(K_{n+1},x) - (n+1)x - nx^2) \\ &= (x+1)D(B_n^c - v,x) - D(K_{n+1},x) + x(1+x)(1+n(1+x)) \\ &= (x+1)D(B_n^c - v,x) - ((1+x)^{n+1}-1) + x(1+x)(1+n(1+x)), \end{aligned}$$

where $(B_n^c/v) \cong B_n^c - v$.

Now, we use Theorem 2.3 to obtain the domination polynomial of the graph $B_n^c - v$. We have

$$\begin{aligned} D(B_n^c - v,x) &= xD(B_n^c - v/u,x) + D((B_n^c - v) - u,x) \\ &+ xD((B_n^c - v) - N[u],x) - (1+x)p_u(B_n^c - v,x). \end{aligned}$$

Since $(B_n^c - v/u) \cong (B_n^c - v) - u \cong CP(n)$ and using Lemma 2.2, we have

$$\begin{aligned} D(B_n^c - v,x) &= (x+1)D(CP(n),x) + x(D(K_n,x)) - (1+x)(D(K_n,x) - nx) \\ &= (x+1)D(CP(n),x) - D(K_n,x) + nx(1+x) \\ &= (x+1)((1+x)^{2n} - (1+2nx)) - ((1+x)^n - 1) + nx(1+x) \\ &= (1+x)^n((1+x)^{n+1}-1) - nx(1+x) - x. \end{aligned}$$

Consequently,

$$\begin{aligned} D(B_n^c,x) &= (x+1)((1+x)^n((1+x)^{n+1}-1) - nx(1+x) - x) \\ &-((1+x)^{n+1}-1) + x(1+x)(1+n(1+x)) \\ &= ((1+x)^{n+1}-1)^2. \end{aligned}$$

The n-barbell graph is the graph on $2n$ vertices which is formed by joining two copies of a complete graph K_n by a single edge, known as a bridge, shown in Fig. 3. We denote this graph by Bar_n. For this graph, we shall

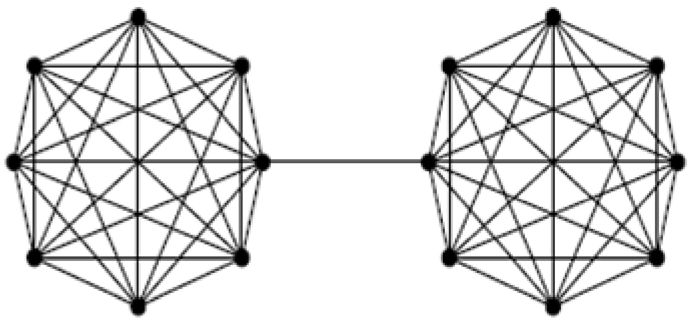

Fig. 3. The barbell graph of order 16, Bar_8

calculate this domination polynomial. We need the following definition and theorems.

An irrelevant edge is an edge $e \in E(G)$, such that $D(G,x) = D(G-e,x)$, and a vertex $v \in V(G)$ is domination-covered, if every dominating set of $G-v$ includes at least one vertex adjacent to v in G [14]. We need the following theorems to obtain the domination polynomial of barbell graph Bar_n.

Theorem 2.5 [14] *Let $G=(V,E)$ be a graph. An vertex $v \in V$ of G is domination-covered if and only if there is a vertex $u \in N[v]$ such that $N[u] \subseteq N[v]$.*

Theorem 2.6 [14] *Let $G=(V,E)$ be a graph. An edge $e=\{u,v\} \in E$ is an irrelevant edge in G, if and only if u and v are domination-covered in $G-e$.*

Theorem 2.7 *For every $n \geq 2$ and $n \in \mathbb{N}$,*

$$D(Bar_n, x) = ((1+x)^n - 1)^2.$$

Proof. Let e be an edge joining two K_n in barbell graph. By Theorem 2.5 two end vertices of edge e are domination-covered in $Bar_n - e$. So, by Theorem 2.6 the edge e is an irrelevant edge of Bar_n. Therefore

$$D(Bar_n, x) = D(Bar_n - e, x) = D(K_n \cup K_n, x) = ((1+x)^n - 1)^2.$$

The following corollary is an immediate consequence of Theorems 2.4 and 2.7.

Corollary 2.8 *For each natural number n, Bar_n and B_{n-1}^c have the same domination polynomial. More precisely, for every n, $[Bar_n] \supseteq \{Bar_n, B_{n-1}^c, K_n \cup K_n\}$, and $[B_{n-1}^c] \supseteq \{Bar_n, B_{n-1}^c, K_n \cup K_n\}$.*

Here, we present some other families of graphs whose are in the $[Bar_n]$. Let define the generalized barbell graphs. As we know, the Bar_n is formed by joining two copies of a complete graph K_n by a single edge. We like to join two copies with more edges as follows:

Definition 2.9 *Suppose that $\{u_1, ..., u_n\}$ and $\{v_1, ..., v_n\}$ are the vertices of two copies of complete graph of order n, K_n and K_n. The generalized barbell graph is denoted by $Bar_{n,t}$ and is a graph with $V(Bar_{n,t}) = \{u_1, ..., u_n\} \cup \{v_1, ..., v_n\}$ and*

$$E(Bar_{n,t}) = E(K_n) \cup E(\mathsf{K}_n) \cup \{u_i v_j \mid 1 \leq i \leq n-1, 1 \leq j \leq n-1\},$$

where $|\{u_i v_j \mid 1 \leq i \leq n-1, 1 \leq j \leq n-1\}| = t$.

As examples see two non-isomorphic graphs $Bar_{3,2}$ in Fig. 4. Notice that B_{n-1}^c is one of the specific case of $B_{n,(n-1)(n-2)}$. The left graph in Fig. 4, is B_2^c.

We have the following theorem.

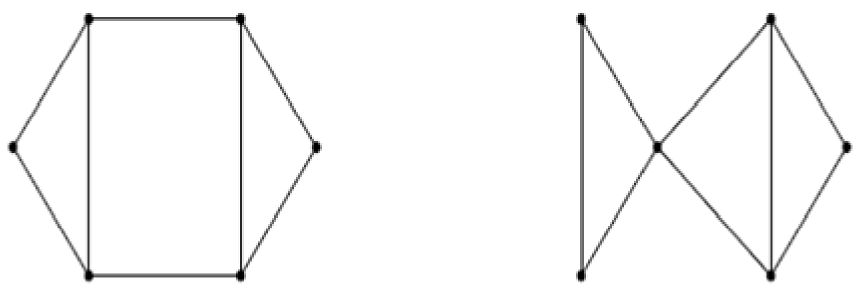

Fig. 4. Two generalized barbell graphs $Bar_{3,2}$

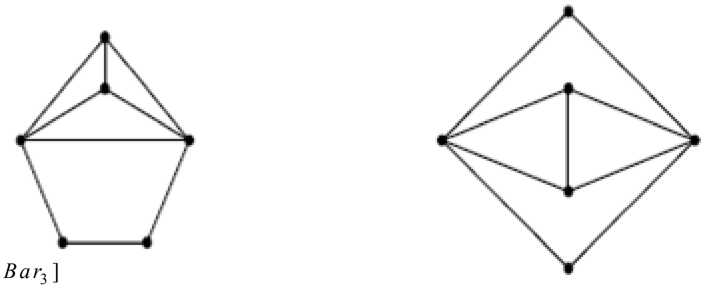

Fig. 5. Two graphs in $[Bar_3]$

Theorem 2.10 *For every* $n \geq 3$ *and* $t \in \mathbb{N}$,

$$D(Bar_{n,t}, x) = ((1+x)^n - 1)^2.$$

Proof. We prove this Theorem by induction on *t*. Suppose that $t = 1$, Then by Theorem 2.7, the result holds. Assume that the result holds for $t = (n-1)^2 - 1$. Let $t = (n-1)^2$ and *e* be the additional edge of $Bar_{n,t}$ to the $Bar_{n,t-1}$. By Theorem 2.5 two end vertices of edge *e* are domination-covered in $Bar_n - e$. So, by Theorem 2.6 the edge e is an irrelevant edge of $Bar_{n,t-1}$. Therefore by the induction hypothesis we have the result.

The following corollary is an immediate consequence of Theorems 2.7 and 2.10.

Corollary 2.11 *For each natural number n and* $t \leq (n-1)^2$, Bar_n *and* $Bar_{n,t}$ *have the same domination polynomial.*

The following example shows that, except for the generalized barbell graphs, there are other graphs in D-equivalence classes of Bar_n.

Example 2.12 *All connected graphs in* $[Bar_3]$ *are the graphs* $Bar_3, Bar_{3,2}, Bar_{3,3}, Bar_{3,4}$ *and two graphs in Fig. 5.*

3. Some Graphs in $[K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}]$

We observed that, for each natural number *n* and $t \leq (n-1)^2$, the domination polynomials of Bar_n and $Bar_{n,t}$ is $((1+x)^n - 1)^2$. In this section, we present graphs whose domination polynomials are $\prod_{i=1}^{k}((1+x)^{n_i} - 1)$. For this purpose, we construct families of graphs from a path P_k which we denote by $S(G_1, G_2, ..., G_k)$ in the following definition.

Definition 3.1 *The graph* $S(G_1, G_2, ..., G_k)$ *is a graph which obtain from a path* P_k *with the vertices* $\{v_1, v_2, \ldots, v_k\}$, *by substituting a graph* G_i *of order* $n_i \geq 3$, *for every vertex* v_i *of* P_k, *such that*

• for $i = 1, k$, the graphs G_i have at least one vertex of degree $n_i - 1$ and other G_i 's have at least two vertices of degree $n_i - 1$, and

• in the graph $S(G_1, G_2, ..., G_k)$, the end vertices of each edge e_i in the path graph, P_k are one vertex of degree $n_i - 1$ in graphs G_{i-1} and G_i.

We have the following result for graph $S(G_1, G_2, ..., G_k)$.

Theorem 3.2 *For every natural number* $k \geq 2$,

$$D(S(G_1,G_2,...,G_k),x)=D(G_1,x)D(G_2,x)\dots D(G_k,x).$$

In particular if $G_i=K_{n_i}$ and $n_i\geq 3$, then

$$D(S(K_{n_1},K_{n_2},...,K_{n_k}),x)=\prod_{i=1}^{k}D(K_{n_i},x)=\prod_{i=1}^{k}((1+x)^{n_i}-1).$$

Proof. Let e_i $(1\leq i\leq k)$ be the edge joining G_{i-1} and G_i in $S(G_1,G_2,...,G_k)$. By Theorem 2.5 two end vertices of edge e_i are domination-covered in $S(G_1,G_2,...,G_k)-e_i$. So, by Theorem 2.6 every edge e_i is an irrelevant edge of $S(G_1,G_2,...,G_k)$. Therefore we have the result.

We shall generalize the graphs $S(G_1,G_2,...,G_k)$ in Definition 3.1 such that this generalized graphs and $S(G_1,G_2,...,G_k)$ have the same domination polynomial. Suppose that $GS_t(K_{n_1},K_{n_2},...,K_{n_k})$ be a family of graphs in the form of $S(K_{n_1},K_{n_2},...,K_{n_k})$ such that the complete graphs K_{n_i} with $V(K_{n_i})=\{u_1,...,u_{n_i}\}$ and $K_{n_{i+1}}$ with $V(K_{n_{i+1}})=\{v_1,...,v_{n_{i+1}}\}$ are joined with t_i following edges

$$\{u_iv_j \mid 1\leq i\leq n_i-1, 1\leq j\leq n_{i+1}-1\},$$

and $t=\sum_{i=1}^{k-1}t_i$. Note that for every $1\leq i\leq k-1$, $t_i\leq(n_i-1)(n_{i+1}-1)$, so $t\leq\sum_{i=1}^{k-1}(n_i-1)(n_{i+1}-1)$. Similar to the proof of the Theorem 2.10, we have the following theorem:

Theorem 3.3 *Let* $n_1,n_2,...,n_k$ *be arbitrary natural numbers and* t_i *be a natural number such that for every* $1\leq i\leq k-1$, $t_i\leq(n_i-1)(n_{i+1}-1)$, $t=\sum_{i=1}^{k-1}t_i$ *and . All graphs in the family of* $GS_t(K_{n_1},K_{n_2},...,K_{n_k})$ *have the same domination polynomial. More precisely, the domination polynomial of each* H *in* $GS_t(K_{n_1},K_{n_2},...,K_{n_k})$ *is equal to* $\prod_{i=1}^{k}((1+x)^{n_i}-1)$.

Conclusion. In this paper, we studied the D-equivalence classes of barbell graphs Bar_n. We showed that, for each natural number *n*, $K_n\cup K_n$, Bar_n, $Bar_{n,t}$ and the complement of the book graph of order $n-1$, B_{n-1}^c have the same domination polynomial, i.e., $[Bar_n]=[Bar_{n,t}]=[B_{n-1}^c]=[K_n\cup K_n]$. Example 2.12, implies that except for these kind of graphs, there are other graphs in this class. Therefore, exact characterization of graphs in $[Bar_n]$ remains as an open problem. Also we presented many families of graphs which are in $[K_{n_1}\cup K_{n_2}\cup...\cup K_{n_k}]$, but similar to Example 2.12, there are other graphs in this class. So, exact characterization of graphs in $[K_{n_1}\cup K_{n_2}\cup...\cup K_{n_k}]$ remains as another open problem.

მათემატიკა

ზოგიერთი გრაფის D-ეკვივალენტობის კლასების შესახებ

ს. ჯაჰარი და ს. ალიხანი

იაზდის უნივერსიტეტი, მათემატიკის დეპარტამენტი, იაზდი, ირანი

(წარმოდგენილია აკადემიის წევრის ნ. ბერიკაშვილის მიერ)

თუ G არის n-რიგის მარტივი გრაფი, მაშინ მისი დომინაციის პოლინომი არის $D(G,x)=\sum d(G,i)x^i$ **, სადაც** $D(G,i)$ **არის G-ს დომინალური სიმრავლეების რაოდენობა სიდიდით i . n-ბარბელ გრაფი** $\mathbf{Bar_n}$ **აგებულია სრული n-რიგის** $\mathbf{K_n}$ **გრაფის ორი ასლის 1-განზომილებიანი წიბოთი შეერთებით. ჩვენ ვამტკიცებთ რომ ყოველი** $n\geq 2$**-თვის,** $\mathbf{Bar_n}$ **არ არის D-ერთადერთი, ე.ი. არსებობს სხვა არაიზომორფული გრაფი იმავე დომინაციის პოლინომით. უფრო ზუსტად, ჩვენ ვაჩვენებთ, რომ ყოველი n-თვის D-ეკვივალენტობის კლასი ბარბელ გრაფისა** $[\mathbf{Bar_n}]$ **შეიცავს მრავალ გრაფს და, მათ შორის, არის n-1 რიგის ე.წ. წიგნის გრაფის დამატება** $B_{n-1}^{c\,\circ}$ **. ჩვენ აგრეთვე აღვწერთ** $K_{n1}\cup K_{n2}\cup\ldots\cup K_{nk}$ **გრაფის D-ეკვივალენტობის კლასში გრაფთა მრავალ ოჯახს.**

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