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Mathematics

On the Asymptotic Estimation of the Generalized Cesàro Means

Teimuraz Akhobadze[*] and Giorgi Gognadze[*]

* *Ivane Javakhishvili Tbilisi State University, Tbilisi*

(Presented by Academy Member Vakhtang Kokilashvili)

ABSTRACT. The behaviour of generalized Cesàro means of trigonometric Fourier series in the space of continuous functions is studied. In particular, deviation of these means from the corresponding continuous functions is established. *© 2016 Bull. Georg. Natl. Acad. Sci.*

Key words: modulus of continuity, Fourier series, generalized Cesàro means

Let f be 2π -periodic continuous function and $S_n(f,x)$ be partial sums of its trigonometric Fourier series:

$$S_n(f,x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k sinkx). \tag{1}$$

Let $C([0,2\pi])$ denotes the space of 2π -periodic continuous functions with the norm.

$$\|f\|_C=\max_{x\in[0,2\pi]}|f(x)|.$$

If $f\in C([0,2\pi])$ then

$$\omega(f,\delta)=\max\{|f(x_1)-f(x_2)|:|x_1-x_2|\le\delta,\ x_1,x_2\in[0,2\pi]\}$$

is called the modulus of continuity of the function f .

If a modulus of continuity ω (see [1]) is given then H^{ω} denotes the class of functions $f\in C([0,2\pi])$ for which

$$\omega(f,\delta)\le\omega(\delta),\ \delta\in[0,2\pi).$$

If $\omega(\delta)=\delta,$ then $H^{\omega}\equiv Lip1$.

Let (α_n) and (S_n) be a sequence of real numbers, where $\alpha_n>-1,\ n\in\mathbb{N}$, and

$$\sigma_n^{\alpha_n}\equiv\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha_n-1}S_\nu\ /\ A_n^{\alpha_n}, \tag{2}$$

where

$$A_k^{\alpha_n}=(\alpha_n+1)(\alpha_n+2)...(\alpha_n+k)/k!.$$

It is clear that $\sigma_n^0=S_n$. If (α_n) is a constant sequence $(\alpha_n=\alpha,\ n\in\mathbb{N})$ then $\sigma_n^{\alpha_n}$ coincides with the usual Cesàro σ_n^{α} - means [2, Ch. III]. If in (2) instead of S_v we substitute $S_v(f,x)$ (see (1)) then the corresponding generalized means $\sigma_n^{\alpha_n}$ is denoted by $\sigma_n^{\alpha_n}(f,x)$.

These means were studied by Kaplan [3]. The author compared the methods of summability (C,α_n) and (C,α), and obtained necessary and sufficient conditions, in terms of the α_n, for the inclusion $(C,\alpha_n)\subset(C,\alpha)$, and sufficient conditions for $(C,\alpha)\subset(C,\alpha_n)$. Later Akhobadze [4-7] and Tetunashvili [8-13] investigated problems of (C,α_n) summability of trigonometric Fourier series.

In [6] Akhobadze among others proved

Theorem 1. *If $f\in H^{\omega}$ and $\alpha_n\in(0,1),\ n=3,4,...,$ then*

$$\|\sigma_n^{-\alpha_n}(f,.)-f(\cdot)\|_C\le C_{\omega}\omega(1/n)\frac{n^{\alpha_n}-1}{\alpha_n(1-\alpha_n)},\tag{3}$$

where C_{ω} is a positive constant depending only on ω.

For $Lip1$ class of functions in the case $\liminf_{n\to\infty}\alpha_n>0$ more precise estimation can be received than (3). Nikol'skii [1] and Stepanetz [14] studied the behaviour of value $\sup_{f\in H^{\omega}}\|S_n(f,.)-f(\cdot)\|_C$. In 1988 Stepanetz raised the question to investigate the order of magnitude of

$$E_n(H^{\omega};\sigma_n^{\alpha_n})\equiv\sup_{f\in H^{\omega}}\|\sigma_n^{\alpha_n}(f,.)-f(\cdot)\|_C\tag{4}$$

for negative constant $(\alpha_n=\alpha,\ n\in\mathbb{N})$ sequence. Akhobadze [15] solved this problem.

Theorem 2. *For every modulus of continuity ω and for every α $(-1<\alpha<0)$*

$$E_n(H^{\omega};\sigma_n^{\alpha})\le\frac{4\Gamma(\alpha+1)}{\pi(2n+1+\alpha)}\sum_{k=0}^{n-1}\left(2\sin\frac{k+1+\alpha/2}{2n+1+\alpha}\pi\right)^{-1-\alpha}\times$$
$$\int_0^{\pi/2}\omega\left(\frac{4t}{2n+1+\alpha}\right)\sin t\,dt+O\left(\frac{1}{n}\int_{1/n}^{\pi}\frac{\omega(t)}{t^2}dt\right).\tag{5}$$

If ω is a convex modulus of continuity then in (5) the inequality can be replaced by equality.

The main object of this article is to investigate asymptotic behaviour of (4) for general sequence α_n $(-1<\alpha_n<0)$.

Theorem 3. *Let ω be an arbitrary modulus of continuity and α_n $(-1<\alpha_n<0)$ be any sequence of numbers. Then*

$$E_n(H^{\omega};\sigma_n^{\alpha_n})\le\frac{4}{\pi(2n+1+\alpha_n)A_n^{\alpha_n}}\sum_{k=0}^{n-1}\left(2\sin\frac{k+1+\alpha_n/2}{2n+1+\alpha_n}\pi\right)^{-1-\alpha_n}\times$$
$$\int_0^{\pi/2}\omega\left(\frac{4t}{2n+1+\alpha_n}\right)\sin t\,dt+O\left(\Gamma(1+\alpha_n)\omega\left(\frac{1}{n}\right)\right)+O\left(\frac{|\alpha_n|}{n}\int_{1/n}^{\pi}\frac{\omega(t)}{t^2}dt\right).\tag{6}$$

if ω is a convex modulus of continuity then in (6) *the inequality can be replaced by equality.*

It is easy to see that Theorem 2 follows from Theorem 3.

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თ. ახობაძე და გ. გოგნაძე

ივანე ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ვ. კოკილაშვილის მიერ)

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REFERENCES:

1. *Nikol'skii S.M.* (1946) Dokl. Akad.Nauk SSSR. 53: 191-197 (in Russian).
2. *Zygmund A.* (1959) Trigonometric Series. Vol. 1, Cambridge University Press.
3. *Kaplan I.* (1960) Izv. Vyssh. Uchebn. Zaved. Mat. **18**, 2: 62-73 (in Russian).
4. *Akhobadze T.* (2004) Bull. Georgian Acad. Sci. **170**, 1: 23-24.
5. *Akhobadze T.* (2007) Acta Math. Hungar. **115** (1-2): 59-78.
6. *Akhobadze T.* (2007) Acta Math. Hungar. **115** (1-2): 79-100.
7. *Akhobadze T.* (2011) Acta Math. Hungar. **130,** 3: 286-308.
8. *Tetunashvili Sh.* (2005) Proc. A. Razmadze Math. Inst. 139: 142-144.
9. *Tetunashvili Sh.* (2007) Proc. A. Razmadze Math. Inst. 145: 130-131.
10. *Tetunashvili Sh.* (2008) Proc. A. Razmadze Math. Inst. 148: 141-145.
11. *Tetunashvili Sh.* (2009) Proc. A. Razmadze Math. Inst. 150: 150-152.
12. *Tetunashvili Sh.* (2011) Proc. A. Razmadze Math. Inst. 155: 130-131.
13. *Tetunashvili Sh.* (2012) Journal of Function Spaces and Applications. 2012:1-9.
14. *Stepanetz A.I.* (1981) Uniform approximations by trigonometric polynomials. Kiev, "Naukova Dumka" (in Russian).
15. *Akhobadze T.* (2004) Bull. Georgian Acad. Sci. **134**, 3: 61-63.

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