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Mathematics

On Regular Cohomologies of Biparabolic Subalgebras of *sl*(*n*)

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ABSTRACT. It is proved that if P is a biparabolic subalgebra of the special linear Lie algebra $sl(n)$ over the field of complex numbers and Z(P) is its center, then $H^n(P,P) = H^n(P,Z(P)),\ n \geq 0$; if P is an indecomposable biparabolic subalgebra, i. e. for corresponding two partitions $(a_1, a_2, ..., a_r)$ and $(b_1, b_2, ..., b_s)$ of *n* partial sums of this partitions never equal each other then $Z(P) = 0$ and, consequently, $H^n(P,P) = 0,\ \ n \geq 0$. Analogous results, for Borel and parabolic subalgebras of semisimple Lie algebras respectively, were obtained by G. Leger, E. Luks [1972] and A. Tolpygo [1972]. *© 2016 Bull. Georg. Natl. Acad. Sci.*

Key words: biparabolic subalgebra, regular representation, Lie algebra cohomologies

Biparabolic Lie subalgebras [1] (initially named “seaweed algebras”) constitute a relatively new object in Lie theory; they generalize the notion of a parabolic subalgebra [2]. There are many articles about cohomologies of parabolic subalgebras and some of their subalgebras, e.g. nilpotent, Heisenberg subalgebras [3-6], but cohomologies of biparabolic subalgebras are not investigated yet. In this paper we investigate regular cohomologies of biparabolic subalgebras of the simple Lie algebras $sl(n)$.

In 1972 Leger and Luks proved that cohomologies of a Borel subalgebra with coefficients in itself (i.e. regular cohomologies) are equal to zero in all dimensions. In the same year Tolpygo proved that this result is true in a more general case, for parabolic subalgebras. We prove that the foresaid result is true for biparabolic subalgebras too, but we consider biparabolic subalgebras only of $sl(n)$; we hope, that this is true also for biparabolic subalgebras of all semisimple Lie algebras.

A biparabolic subalgebra [1] of the special linear Lie algebra $sl(n)$ over the field of complex numbers is the intersection of two parabolic subalgebras of $sl(n)$ whose sum is $sl(n)$; we may represent such subalgebra graphically as

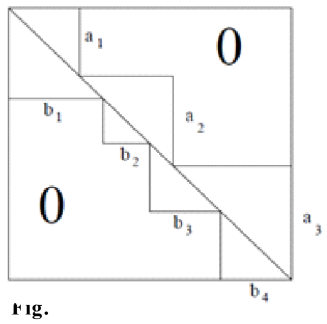

Fig.

It is apparent, that two partitions of n exist.

$$(a_1, a_2, ..., a_r),\ (b_1, b_2, ..., b_s) \tag{1}$$

(i.e. $\sum a_i = \sum b_i = n$, a_i and b_i are natural numbers) such that from $a_{ij} \neq 0$ it follows that $a_t + 1 \leq a_{ij} \leq a_{t+1}$ if $i \leq j$ and $b_t + 1 \leq a_{ij} \leq b_{t+1}$ if $i \geq j$.

Our main results are:

Theorem 1. *If P is a biparabolic subalgebra of $sl(n)$ and $Z(P)$ is its center, then for all $n \geq 0$*

$$H^n(P,P) \cong H^n(P, Z(P)).$$

If partial sums of a pair of partitions never equal each other, i.e.

$$a_1 + a_2 + ... + a_{r_1} \neq b_1 + b_2 + ... + b_{s_1},$$

where $r_1 < r$, $s_1 < s$, then we call such a pair *indecomposable*.

Theorem 2. *If the pair of partitions corresponding to the biparabolic subalgebra P of $sl(n)$ is indecomposable, then*

$$H^n(P,P) = 0.$$

To prove these theorems, we need to describe our main object more accurately. Let R denote the reductive subalgebra of P (R consists of block-diagonal submatrices of relevant shape in $sl(n)$) and let $N = N_1 + N_2$ denote the nilradical of *P*; there, N_1 is located above R and, N_2 is located below *R*. It is clear, that $P \cong R + N$ as vector spaces. The Cartan subalgebra *H* of *P* coincides with the diagonal of *P*and

$$sl(n) \cong (R + N_1 + N_2) + \overline{N_1} + \overline{N_2} + M + \overline{M}, \tag{2}$$

as vector spaces; here $\overline{N_1}$ and $\overline{N_2}$ are conjugated by the Killing form to N_1 and N_2 respectively, M is a top right supplement of $P + N_1 + \overline{N_2}$ in $sl(n)$ and $\overline{M}$ is conjugated by the Killing form to M.

Lemma 1.

$$H^i(N,P)^R = \begin{cases} Z(P), & \text{if } i = 0 \\ 0, & \text{if } i > 0. \end{cases}$$

Sketch of proof. If we choose conjugated by Killing form bases $\{u_i\}$ and $\{u^i\}$ in $sl(n)$ as in [4], then we may construct a homotopy operator

$$C^i(sl(n), sl(n)) \xrightarrow{k} C^{i-1}(sl(n), sl(n))$$

by the formula

$$(kf)(g_1, g_2, ..., g_{i-1}) = \sum u^i f(u_i, g_1, g_2, ..., g_{i-1}).$$

Let us consider the chain of maps:

$$C^i(N,P) \xrightarrow{\psi} C^i(sl(n), sl(n)) \xrightarrow{d} C^{i-1}(sl(n), sl(n)) \xrightarrow{k} C^i(sl(n), sl(n)) \xrightarrow{\phi} C^i(N, sl(n));$$

there, ψ is induced by the projection $sl(n) \to N$ – see (2) – and ϕ is induced by restriction on N. As in [4] we can prove that this map induces in dimensions $i \geq 1$ injections

$$Z(C^i(N,P)^R) \to B(C^i(N,P)^R),$$

i.e., in this case $H^i(N,P) = 0$. The case $i = 0$ is proved by direct computations.

Lemma 2. *If* P *is indecomposable, then* $Z(P) = 0$.

Lemma 2 is proved by induction with respect to the sum $r + s$ (see [1]).

Let us now prove theorem 1. Since $R \cong P/N$, we can construct a spectral sequence

$$E_2^{i,j} = H^i(R, H^j(N,P)) \Rightarrow H^n(P,V).$$

It is well known [7] that if V is a semisimple module over a reductive Lie algebra R then

$$H^i(R,V) = H^i(R, V^R).$$

Therefore, our statement follows from Lemma 1.

As for Theorem 2, it follows from Theorem 1 and Lemma 2.

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მათემატიკა

sl(*n*)-ის ბიპარაბოლური ქვეალგებრების რეგულარული კოჰომოლოგიების შესახებ

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ნაშრომში დამტკიცებულია, რომ თუ P არის კომპლექსურ რიცხვთა ველზე განსაზღვრული სპეციალური წრფივი $sl(n)$ ლის ალგებრის ბიპარაბოლური ქვეალგებრა და $Z(P)$ არის მისი ცენტრი, მაშინ $H^n(P,P) = H^n(P, Z(P)),\ n \geq 0.$ თუ P არის დაუშლადი ბიპარაბოლური ქვეალგებრა, ე.ი.

n-ის შესაბამისი $(a_1, a_2, ..., a_r)$ და $(b_1, b_2, ..., b_s)$ დაშლებისთვის ნაწილობრივი ჯამები არასოდეს არის ერთმანეთის ტოლი, მაშინ $Z(P) = 0$. ამ პირობებში P-ს რეგულარული კოჰომოლოგიები უდრის ნულს: $H^n(P, P) = 0,\ n \geq 0.$

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