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Mathematics

Multiplicative Versions of Zagreb Indices under Subdivision Operators

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ABSTRACT. In this paper, we compare the multiplicative versions of Zagreb indices under the subdivision operators L, S, R, Q and T. Results are applied to obtain several interesting inequalities for the multiplicative Zagreb indices and multiplicative-sum Zagreb index of these operators in terms of the order, size, first Zagreb index, first and second multiplicative Zagreb indices and multiplicative-sum Zagreb index of the primary graph. *© 2016 Bull. Georg. Natl. Acad. Sci.*

Key words: degree; lower bound; upper bound; multiplicative Zagreb indices; multiplicative-sum Zagreb index; subdivision operators.

1. Introduction

Throughout the paper, we consider connected finite graphs without any loops or multiple edges. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $V(e)$ the set of two end vertices of the edge e of G, and by $d_G(u)$ the degree of the vertex u in G which is the number of edges incident to u.

In theoretical chemistry, the physico–chemical properties of chemical compounds are often modeled by means of molecular–graph–based structure–descriptors, which are also referred to as *topological indices* [1, 2]. The *Zagreb indices* [3] are among the oldest topological indices, and were introduced as early as in 1972. These indices have since been used to study molecular complexity, chirality, ZE–isomerism and hetero–systems. For details on their theory and applications see the recent papers [4–9]. The first and second Zagreb indices of a graph G are denoted by $M_1(G)$ and $M_2(G)$, respectively and defined as

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 \quad \text{and} \quad M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

The first Zagreb index can also be expressed as a sum over edges of G:

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u)+d_G(v)].$$

The multiplicative versions of Zagreb indices were introduced by Todeschini and Consonni [10] in 2010. The first and second *multiplicative Zagreb indices* of a graph G are denoted by $\Pi_1(G)$ and $\Pi_2(G)$, respectively and defined as

$$\Pi_1(G) = \prod_{u \in V(G)} d_G(u)^2 \quad \text{and} \quad \Pi_2(G) = \prod_{uv \in E(G)} d_G(u) d_G(v) .$$

The second multiplicative Zagreb index can also be expressed as a product over vertices of G [11]:

$$\Pi_2(G) = \prod_{u \in V(G)} d_G(u)^{d_G(u)} .$$

In 2012, Eliasi et al. [12] introduced another multiplicative version of the first Zagreb index called *multiplicative-sum Zagreb index*. The multiplicative-sum Zagreb index of a graph G is denoted by $\Pi_1^*(G)$ and defined as

$$\Pi_1^*(G) = \prod_{uv \in E(G)} [d_G(u) + d_G(v)] .$$

We refer the reader to [13–17] for mathematical properties and applications of the multiplicative Zagreb indices and multiplicative-sum Zagreb index.

It is well known that many graphs of general and in particular of chemical interest arise from simpler graphs via various *graph operators*. It is, hence, important to understand how certain invariants of a graph change under graph operators. In this paper, we study the behavior of the first and second multiplicative Zagreb indices and multiplicative-sum Zagreb index under subdivision operators.

2. Definitions and Preliminaries

In this section, we recall the definitions of subdivision-related graphs and state some preliminary results about them.

Let $G = (V(G), E(G))$ be a nontrivial simple connected graph with vertex set $V(G)$ and edge set $E(G)$ and let $|V(G)| = n$ and $|E(G)| = m$. Related to the graph G, the *line graph* $L(G)$, the *subdivision graph* $S(G)$, and the *total graph* $T(G)$ are defined as follows.

The line graph $L(G)$ is the graph whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common.

The subdivision graph $S(G)$ is the graph obtained from G by replacing each of its edges by a path of length two, or equivalently, by inserting an additional vertex into each edge of G.

The total graph $T(G)$ is the graph whose vertex set is $V(G) \bigcup E(G)$, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident.

Two extra subdivision operators named $R(G)$ and $Q(G)$ are defined as follows.

$R(G)$ is the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it.

$Q(G)$ is the graph obtained from G by inserting a new vertex into each edge of G and by joining with edges those pairs of these new vertices which lie on adjacent edges of G.

Now consider the sets $EE(G)$ and $EV(G)$ for the graph $G = (V(G), E(G))$ as follows.

$$EE(G) = \{ee' \mid e, e' \in E(G), |V(e) \bigcap V(e')| = 1\} , \quad EV(G) = \{ev \mid e \in E(G), v \in V(e)\} .$$

It is easy to see that,

$$|EE(G)| = \sum_{u \in V(G)} \binom{d_G(u)}{2} = \frac{1}{2} M_1(G) - m\,, \ |EV(G)| = 2m\,.$$

Based on the definitions of these sets, we may write the subdivision-related graphs as follows.

$$L(G) = (E(G), EE(G))\,,$$
$$S(G) = (V(G) \cup E(G), EV(G))\,,$$
$$R(G) = (V(G) \cup E(G), E(G) \cup EV(G))\,,$$
$$Q(G) = (V(G) \cup E(G), EE(G) \cup EV(G))\,,$$
$$T(G) = (V(G) \cup E(G), E(G) \cup EE(G) \cup EV(G))\,.$$

Obviously,

$$|V(L(G))| = m\,, \ |V(S(G))| = |V(R(G))| = |V(Q(G))| = |V(T(G))| = n + m\,,$$

and

$$|E(S(G))| = 2m\,, \ |E(R(G))| = 3m\,,$$

$$|E(L(G))| = \frac{1}{2} M_1(G) - m\,, \ |E(Q(G))| = \frac{1}{2} M_1(G) + m\,, \ |E(T(G))| = \frac{1}{2} M_1(G) + 2m\,.$$

In the following lemma, we find the relationship among the degree of vertices in subdivision-related graphs.

Lemma 1. *For any vertex* $v \in V(G)$,

$$d_{R(G)}(v) = d_{T(G)}(v) = 2d_{S(G)}(v) = 2d_{Q(G)}(v) = 2d_G(v)\,,$$

and for any edge $e = uv \in E(G)$,

$$d_{S(G)}(e) = d_{R(G)}(e) = 2\,, \quad d_{Q(G)}(e) = d_{T(G)}(e) = d_{L(G)}(e) + 2 = d_G(u) + d_G(v)\,.$$

Proof. By definition of the subdivision-related graphs the proof is obvious.

3. Main Results

In this section, we study the behavior of the first and second multiplicative Zagreb indices and multiplicative-sum Zagreb index under subdivision operators. Throughout this section, let G be a nontrivial simple connected graph with vertex set $V(G)$ and edge set $E(G)$ and let n and m denote its order and size, respectively. We start by computing the first multiplicative Zagreb index of the subdivision operators S, R, Q and T.

Theorem 1. *Let* G *be a graph of order* n *and size* m. *Then*

(i) $\Pi_1(S(G)) = 4^m \Pi_1(G)$,

(ii) $\Pi_1(R(G)) = 4^{n+m} \Pi_1(G)$,

(iii) $\Pi_1(Q(G)) = \Pi_1(G) \Pi_1^*(G)^2$,

(iv) $\Pi_1(T(G)) = 4^n \Pi_1(G) \Pi_1^*(G)^2$.

Proof. *(i)* By definition of $S(G)$ and Lemma 1,

$$\Pi_1(S(G)) = \prod_{u \in V(G)} d_G(u)^2 \times \prod_{e \in E(G)} 2^2 = 4^m \Pi_1(G)\,.$$

(ii) By definition of $R(G)$ and Lemma 1,

$$\Pi_1(R(G)) = \prod_{u\in V(G)} (2d_G(u))^2 \times \prod_{e\in E(G)} 2^2 = 4^n \Pi_1(G)\times 4^m = 4^{n+m}\Pi_1(G).$$

(iii) By definition of $Q(G)$ and Lemma 1,

$$\Pi_1(Q(G)) = \prod_{u\in V(G)} d_G(u)^2 \times \prod_{uv\in E(G)} [d_G(u)+d_G(v)]^2 = \Pi_1(G)\Pi_1^*(G)^2.$$

(iv) By definition of $T(G)$ and Lemma 1,

$$\Pi_1(T(G)) = \prod_{u\in V(G)} (2d_G(u))^2 \times \prod_{uv\in E(G)} [d_G(u)+d_G(v)]^2 = 4^n \Pi_1(G)\Pi_1^*(G)^2.$$

As a direct consequence of Theorem 1, we get the following corollary.

Corollary 1. *Let G be a graph of order n. Then*

$$\frac{\Pi_1(R(G))}{\Pi_1(S(G))} = \frac{\Pi_1(T(G))}{\Pi_1(Q(G))} = 4^n.$$

Using Lemma 1 and AM-GM inequality, we obtain a sharp upper bound for the first multiplicative Zagreb index of the line graph $L(G)$ in terms of the multiplicative-sum Zagreb index and the size of the graph G.

Theorem 2. *Let G be a graph of size m. Then*

$$\Pi_1(L(G)) \le \frac{\Pi_1^*(G)^4}{64^m},$$

with equality if and only if G is a cycle or the star graph on 4 vertices.

Proof. By definition of the multiplicative-sum Zagreb index and Lemma 1,

$$\Pi_1^*(G)^4 = \prod_{uv\in E(G)} [d_G(u)+d_G(v)]^4 = \prod_{uv\in V(L(G))} [d_{L(G)}(uv)+2]^4.$$

Now by AM-GM inequality,

$$\Pi_1^*(G)^4 \ge \prod_{uv\in V(L(G))} \left[2\sqrt{d_{L(G)}(uv)\times 2}\right]^4 = \prod_{uv\in V(L(G))} 64 d_{L(G)}(uv)^2 = 64^m \Pi_1(L(G)).$$

So,

$$\Pi_1(L(G)) \le \frac{\Pi_1^*(G)^4}{64^m}.$$

By AM-GM inequality, the above equality holds if and only if for every $uv\in E(G)$, $d_{L(G)}(uv)=2$. This by Lemma 1 implies that, for every $uv\in E(G)$, $d_G(u)+d_G(v)=4$. So, G is a cycle or the star graph on 4 vertices.

Now, we introduce the quantity $\Gamma(G)$ related to a simple connected graph G as follows.

$$\Gamma(G) = \prod_{uv\in E(G)} [d_G(u)+d_G(v)]^{d_G(u)+d_G(v)}.$$

In the following theorem, we determine the second multiplicative Zagreb index of subdivision operators.

Theorem 3. *Let G be a graph of size m. Then*

(i) $\Pi_2(S(G)) = 4^m \Pi_2(G)$,

(ii) $\Pi_2(R(G)) = 64^m \Pi_2(G)^2$,

(iii) $\Pi_2(Q(G)) = \Pi_2(G)\Gamma(G)$,

(iv) $\Pi_2(T(G)) = 16^m \Pi_2(G)^2 \Gamma(G)$.

Proof. *(i)* By definition of $S(G)$ and Lemma 1,

$$\Pi_2(S(G)) = \prod_{u\in V(G)} d_G(u)^{d_G(u)} \times \prod_{e\in E(G)} 2^2 = 4^m \Pi_2(G).$$

(ii) By definition of $R(G)$ and Lemma 1,

$$\Pi_2(R(G)) = \prod_{u\in V(G)} (2d_G(u))^{2d_G(u)} \times \prod_{e\in E(G)} 2^2 = \prod_{u\in V(G)} 4^{d_G(u)} \times \prod_{u\in V(G)} \left(d_G(u)^{d_G(u)}\right)^2 \times 4^m$$

$$= 4^{2m} \times \Pi_2(G)^2 \times 4^m = 64^m \Pi_2(G)^2.$$

(iii) By definition of $Q(G)$ and Lemma 1,

$$\Pi_2(Q(G)) = \prod_{u\in V(G)} d_G(u)^{d_G(u)} \times \prod_{uv\in E(G)} [d_G(u) + d_G(v)]^{d_G(u)+d_G(v)} = \Pi_2(G)\Gamma(G).$$

(iv) By definition of $T(G)$ and Lemma 1,

$$\Pi_2(T(G)) = \prod_{u\in V(G)} (2d_G(u))^{2d_G(u)} \times \prod_{uv\in E(G)} [d_G(u) + d_G(v)]^{d_G(u)+d_G(v)} = 16^m \Pi_2(G)^2 \Gamma(G).$$

As a direct consequence of Theorem 3, we get the following corollary.

Corollary 2. *Let G be a graph of size m. Then*

$$\frac{\Pi_2(R(G))}{\Pi_2(S(G))} = \frac{\Pi_2(T(G))}{\Pi_2(Q(G))} = 16^m \Pi_2(G).$$

In the following theorem, we obtain a sharp upper bound for the second multiplicative Zagreb index of $L(G)$ in terms of the quantity $\Gamma(G)$, the first Zagreb index, the multiplicative-sum Zagreb index and size of the graph G.

Theorem 4. *Let G be a graph of size m. Then*

$$\Pi_2(L(G)) \le \frac{\Gamma(G)^2}{8^{M_1(G)-2m}\Pi_1^*(G)^4},$$

with equality if and only if G is a cycle or the star graph on 4 vertices.

Proof. By definition of $\Gamma(G)$, we have

$$\Gamma(G)^2 = \prod_{uv\in E(G)} [d_G(u) + d_G(v)]^{2(d_{L(G)}(uv)+2)} = \Pi_1^*(G)^4 \times \prod_{uv\in V(L(G))} [d_{L(G)}(uv) + 2]^{2d_{L(G)}(uv)}.$$

Now by AM-GM inequality,

$$\Gamma(G)^2 \ge \Pi_1^*(G)^4 \times \prod_{uv\in V(L(G))} \left[2\sqrt{d_{L(G)}(uv) \times 2}\right]^{2d_{L(G)}(uv)}$$

$$= \Pi_1^*(G)^4 \times \prod_{uv\in V(L(G))} 8^{d_{L(G)}(uv)} \times \prod_{uv\in V(L(G))} d_{L(G)}(uv)^{d_{L(G)}(uv)}$$

$$= \Pi_1^*(G)^4 \times 8^{\sum_{uv\in V(L(G))} d_{L(G)}(uv)} \times \Pi_2(L(G)) = 8^{M_1(G)-2m}\Pi_1^*(G)^4 \Pi_2(L(G)).$$

So,

$$\Pi_2(L(G)) \leq \frac{\Gamma(G)^2}{8^{M_1(G)-2m}\Pi_1^*(G)^4}.$$

By AM-GM inequality, the above equality holds if and only if for every $uv \in E(G)$, $d_{L(G)}(uv) = 2$, that is $d_G(u) + d_G(v) = 4$. So, G is a cycle or the star graph on 4 vertices.

Using parts *(iii)* and *(iv)* of Theorem 3 and then by Theorem 4, we can obtain a sharp inequality for the second multiplicative Zagreb index of $Q(G)$ and $T(G)$.

Corollary 3. *Let G be a graph of size m. Then*

$$\Pi_2(Q(G)) \geq (2\sqrt{2})^{M_1(G)-2m}\Pi_1^*(G)^2\Pi_2(G)\sqrt{\Pi_2(L(G))},$$

with equality if and only if G is a cycle or the star graph on 4 vertices.

Corollary 4. *Let G be a graph of size m. Then*

$$\Pi_2(T(G)) \geq 2^m(2\sqrt{2})^{M_1(G)}\left(\Pi_1^*(G)\Pi_2(G)\right)^2\sqrt{\Pi_2(L(G))},$$

with equality if and only if G is a cycle or the star graph on 4 vertices.

Now, we turn our attention toward multiplicative-sum Zagreb index of subdivision operators. In the following theorem, we find a formula for the multiplicative-sum Zagreb index of the subdivision graph $S(G)$.

Theorem 5. *Let G be a graph. Then*

$$\Pi_1^*(S(G)) = \prod_{u\in V(G)}[d_G(u)+2]^{d_G(u)}.$$

Proof. By definition of $S(G)$ and Lemma 1,

$$\Pi_1^*(S(G)) = \prod_{ev\in EV(G)}[d_{S(G)}(e)+d_{S(G)}(v)] = \prod_{uv\in E(G)}[d_G(u)+2][d_G(v)+2].$$

The vertex $u \in V(G)$ is the endpoint of $d_G(u)$ edges of G. Therefore in the above product, the factor $d_G(u)+2$ occurs $d_G(u)$ times. So,

$$\Pi_1^*(S(G)) = \prod_{u\in V(G)}[d_G(u)+2]^{d_G(u)}.$$

Using Theorem 5 and AM-GM inequality, we can obtain a sharp lower bound for the multiplicative-sum Zagreb index of $S(G)$ in terms of the second multiplicative Zagreb index and size of the graph G.

Corollary 5. *Let G be a graph of size m. Then*

$$\Pi_1^*(S(G)) \geq 8^m\sqrt{\Pi_2(G)}.$$

with equality if and only if G is a cycle.

Proof. Using Theorem 5 and AM-GM inequality, we have

$$\Pi_1^*(S(G)) = \prod_{u\in V(G)}[d_G(u)+2]^{d_G(u)} \geq \prod_{u\in V(G)}\left[2\sqrt{d_G(u)\times 2}\right]^{d_G(u)}$$

$$= (2\sqrt{2})^{\sum_{u\in V(G)}d_G(u)} \times \sqrt{\prod_{u\in V(G)}d_G(u)^{d_G(u)}} = 8^m\sqrt{\Pi_2(G)}.$$

By AM-GM inequality, the above equality holds if and only if for every $u \in V(G)$, $d_G(u) = 2$. This implies that G is a cycle.

In the following theorem, we find a formula for the multiplicative-sum Zagreb index of $R(G)$.

Theorem 6. *Let G be a graph of size m. Then*

$$\Pi_1^*(R(G)) = 8^m \Pi_1^*(G) \prod_{u\in V(G)} [d_G(u)+1]^{d_G(u)}.$$

Proof. By definition of $R(G)$ and Lemma 1,

$$\Pi_1^*(R(G)) = \prod_{uv\in E(G)} [d_{R(G)}(u)+d_{R(G)}(v)] \times \prod_{ev\in EV(G)} [d_{R(G)}(e)+d_{R(G)}(v)]$$

$$= \prod_{uv\in E(G)} [2d_G(u)+2d_G(v)] \times \prod_{uv\in E(G)} [2d_G(u)+2][2d_G(v)+2]$$

$$= 2^m \prod_{uv\in E(G)} [d_G(u)+d_G(v)] \times 4^m \prod_{uv\in E(G)} [d_G(u)+1][d_G(v)+1]$$

$$= 8^m \Pi_1^*(G) \prod_{u\in V(G)} [d_G(u)+1]^{d_G(u)}.$$

Using Theorem 6 and AM-GM inequality, we can obtain a sharp lower bound for the multiplicative-sum Zagreb index of $R(G)$ in terms of the multiplicative-sum Zagreb index, second multiplicative Zagreb index and size of the graph G.

Corollary 6. *Let G be a graph of size m. Then*

$$\Pi_1^*(R(G)) \geq 32^m \Pi_1^*(G)\sqrt{\Pi_2(G)},$$

with equality if and only if G is the 2-vertex path P_2.

Proof. Using Theorem 6 and AM-GM inequality, we h ave

$$\Pi_1^*(R(G)) = 8^m \Pi_1^*(G) \prod_{u\in V(G)} [d_G(u)+1]^{d_G(u)} \geq 8^m \Pi_1^*(G) \prod_{u\in V(G)} \left[2\sqrt{d_G(u)\times 1}\right]^{d_G(u)}$$

$$= 8^m \Pi_1^*(G) \times 2^{\sum_{u\in V(G)} d_G(u)} \times \sqrt{\prod_{u\in V(G)} d_G(u)^{d_G(u)}} = 32^m \Pi_1^*(G)\sqrt{\Pi_2(G)}.$$

By AM-GM inequality, the above equality holds if and only if for every $u \in V(G)$, $d_G(u) = 1$. So, G is the 2-vertex path P_2.

In order to obtain some lower bounds on the multiplicative-sum Zagreb index of $Q(G)$ and $T(G)$, we need to prove two following lemmas.

Lemma 2. *Let G be a graph of size m. Then*

$$\prod_{ee'\in EE(G)} [d_{Q(G)}(e)+d_{Q(G)}(e')] = \prod_{ee'\in EE(G)} [d_{T(G)}(e)+d_{T(G)}(e')] \geq 2^{M_1(G)-2m}\sqrt{\Pi_1^*(L(G))},$$

with equality if and only if G is a cycle or the star graph on 4 vertices.

Proof. Using Lemma 1, we have

$$\prod_{ee'\in EE(G)} [d_{Q(G)}(e)+d_{Q(G)}(e')] = \prod_{ee'\in EE(G)} [d_{T(G)}(e)+d_{T(G)}(e')]$$

$$= \prod_{ee'\in E(L(G))} [d_{L(G)}(e)+d_{L(G)}(e')+4].$$

Now by AM-GM inequality,

$$\prod_{ee'\in E(L(G))}[d_{L(G)}(e)+d_{L(G)}(e')+4]\geq \prod_{ee'\in E(L(G))}2\sqrt{\left(d_{L(G)}(e)+d_{L(G)}(e')\right)\times 4}$$

$$=4^{|E(L(G))|}\sqrt{\prod_{ee'\in E(L(G))}\left(d_{L(G)}(e)+d_{L(G)}(e')\right)}=2^{M_1(G)-2m}\sqrt{\Pi_1^*(L(G))}\,.$$

By AM-GM inequality, the above equality holds if and only if for every $ee'\in E(L(G))$, $d_{L(G)}(e)+d_{L(G)}(e')=4$. So, for every $uv, zv\in E(G)$,

$$\left(d_G(u)+d_G(v)-2\right)+\left(d_G(z)+d_G(v)-2\right)=4,$$

that is $2d_G(v)+d_G(u)+d_G(z)=8$. This implies that, for every $uv, zv\in E(G)$, $d_G(u)=d_G(v)=d_G(z)=2$ or $d_G(v)=3$, $d_G(u)=d_G(z)=1$. So, G is a cycle or the star graph on 4 vertices.

Lemma 3. *Let G be a graph of size m. Then*

(i) $\prod_{ev\in EV(G)}[d_{Q(G)}(e)+d_{Q(G)}(v)]>(2\sqrt{2})^m\,\Pi_1^*(G)\sqrt{\Pi_2(G)}$,

(ii) $\prod_{ev\in EV(G)}[d_{T(G)}(e)+d_{T(G)}(v)]>(4\sqrt{3})^m\,\Pi_1^*(G)\sqrt{\Pi_2(G)}$.

Proof. *(i)* Using Lemma 1, we have

$$\prod_{ev\in EV(G)}[d_{Q(G)}(e)+d_{Q(G)}(v)]$$

$$=\prod_{uv\in E(G)}\left[d_G(u)+\left(d_G(u)+d_G(v)\right)\right]\left[d_G(v)+\left(d_G(u)+d_G(v)\right)\right]$$

$$=\prod_{uv\in E(G)}\left[2d_G(u)+d_G(v)\right]\left[2d_G(v)+d_G(u)\right]$$

$$=\prod_{uv\in E(G)}\left[2\left(d_G(u)+d_G(v)\right)^2+d_G(u)d_G(v)\right].$$

Now by AM-GM inequality,

$$\prod_{ev\in EV(G)}[d_{Q(G)}(e)+d_{Q(G)}(v)]>\prod_{uv\in E(G)}2\sqrt{2\left(d_G(u)+d_G(v)\right)^2\times d_G(u)d_G(v)}$$

$$=(2\sqrt{2})^m\,\Pi_1^*(G)\sqrt{\Pi_2(G)}\,.$$

Note that, the above inequality is strict. Since by AM-GM inequality the equality holds if and only if for every $uv\in E(G)$, $2\left(d_G(u)+d_G(v)\right)^2=d_G(u)d_G(v)$, which is a contradiction.

(ii) Using the same argument as in the proof of part *(i)*, we can get the desired result.

Now, we apply Lemma 2 and Lemma 3 to obtain lower bounds on the multiplicative-sum Zagreb index of $Q(G)$ and $T(G)$.

Theorem 7. *Let G be a graph of size m. Then*

(i) $\Pi_1^*(Q(G))>(\sqrt{2})^{2M_1(G)-m}\,\Pi_1^*(G)\sqrt{\Pi_2(G)\Pi_1^*(L(G))}$,

(ii) $\Pi_1^*(T(G))>2^{M_1(G)+m}(\sqrt{3})^m\,\Pi_1^*(G)^2\sqrt{\Pi_2(G)\Pi_1^*(L(G))}$.

Proof. *(i)* By definition of the multiplicative-sum Zagreb index, we have

$$\Pi_1^*(Q(G)) = \prod_{ee'\in EE(G)} [d_{Q(G)}(e) + d_{Q(G)}(e')] \times \prod_{ev\in EV(G)} [d_{Q(G)}(e) + d_{Q(G)}(v)].$$

Now using Lemma 2 and Lemma 3, we have

$$\Pi_1^*(Q(G)) > 2^{M_1(G)-2m} \sqrt{\Pi_1^*(L(G))} \times (2\sqrt{2})^m \Pi_1^*(G)\sqrt{\Pi_2(G)}$$

$$= (\sqrt{2})^{2M_1(G)-m} \Pi_1^*(G)\sqrt{\Pi_2(G)\Pi_1^*(L(G))}.$$

(ii) By definition of the multiplicative-sum Zagreb index, we have

$$\Pi_1^*(T(G)) = \prod_{uv\in E(G)} [d_{T(G)}(u) + d_{T(G)}(v)] \times \prod_{ee'\in EE(G)} [d_{T(G)}(e) + d_{T(G)}(e')]$$

$$\times \prod_{ev\in EV(G)} [d_{T(G)}(e) + d_{T(G)}(v)].$$

By Lemma 1,

$$\prod_{uv\in E(G)} [d_{T(G)}(u) + d_{T(G)}(v)] = \prod_{uv\in E(G)} [2d_G(u) + 2d_G(v)] = 2^m \Pi_1^*(G).$$

Now using Lemma 2 and Lemma 3, we have

$$\Pi_1^*(T(G)) > 2^m \Pi_1^*(G) \times 2^{M_1(G)-2m} \sqrt{\Pi_1^*(L(G))} \times (4\sqrt{3})^m \Pi_1^*(G)\sqrt{\Pi_2(G)}$$

$$= 2^{M_1(G)+m} (\sqrt{3})^m \Pi_1^*(G)^2 \sqrt{\Pi_2(G)\Pi_1^*(L(G))}.$$

4. Conclusion

In this paper, we obtain some relationship among the multiplicative versions of Zagreb indices of subdivision operators. The behavior of other vertex-degree-based graph invariants can be studied under subdivision operators by applying the method considered in this paper.

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