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Mathematics

Approximate Solution of Boundary Value Problems for the Ordinary Second-Order Differential Equation with Variable Coefficients by Means of Operator Interpolation Method

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ABSTRACT. New computing algorithms for approximate solution of the two-point boundary value problem with variable coefficients are described in the paper. Green function of the given boundary value problem considered as a non-linear operator with respect to the variable coefficient is approximated by means of operator interpolation polynomial of the Newton type. For approximation of the inverse operator two different types of formulae are constructed. Conventionally these formulas can be called direct and modified formulas. Consequently, for approximate solution of the two-point boundary value problem with variable coefficients direct and modified interpolation operator methods are used. Description of the algorithms for approximate solution are provided and the computation results of the test problems are given in tables.

Key words: two-point boundary value problem, Green function, operator interpolation polynomial of the Newton type.

In the theory of the non-linear systems, for solution of the identification problems the functional series and interpolation polynomials are used [1-3]. In the works of V. Makarov and V. Khlobistov [2, 3], for non-linear functions (operators) an interpolation formula of the Newton type is constructed and the value of the remainder term is received. Such approach is based on defining polynomial kernel of the functional (operator) from interpolation conditions on "continual" units, which represent a linear combination of the Heavyside functions. The above mentioned works have a theoretical and practical significance in applied problems of the approximation theory of operators. The above mentioned authors do not consider the problem of realization of interpolation approximation on electronic processing machines (PCs). In [4, 5], for approximate solution of the boundary problems of elliptic differential equations with variable coefficients the computing algorithms are described, the computation results of the test problems are given, the convergence problems are studied by the numerical-experimental method. In the present work, as well as in [4], the problems approximate solution of the two-point boundary value problem with variable coefficients by means of operator interpolation polynomial of the Newton type are considered. Here, the Green function of differential equation (boundary value problem) as a non-linear operator with respect to the variable coefficient is changed by kernels known with operator interpolation polynomials of Newton type. For approximate solution of the two-point boundary value problem various computational formulae are constructed.The description of the realizing algorithm and the computation results for the test problems are given. From the series of numerical experiments the convergence with respect to *m* parameter (*m* - the degree of operator interpolation polynomial of the Newton type) is revealed.

Problem. Consider the two-point boundary value problem for ordinary differential equation:

$$\begin{cases} u''(x)-q(x)u(x)=-f(x), & x\in[0,1], \\ u(0)=u(1)=0, & q(x)\geq 0,\ q(x),\ f(x)\in L_2[0,1] \end{cases} \tag{1}$$

$$q(x)\geq 0,\ q(x),\ f(x)\in L_2[0,1].$$

Problem (1) in space $W_{2,0}^2(0,1)$ has the only solution, which can be given by the Green's function as follows:

$$u(x)=\int_0^1 G(x,\xi,q(.))\,f(\xi)\,d\xi \tag{2}$$

Denotation $G(x,\xi,q(.))$ shows that the boundary $G(x,\xi)$ depends on the Green's function $q(x)$ (variable coefficient), i.e. Green's function can be considered as a non-linear operator with respect to *q:*

$$G: L_2^+[0,1]\to C([0,1]\times[0,1]),$$

$$L_2^+[0,1]=\{q(x): q(x)\in L_2[0,1], q(x)\geq 0, \forall x\in[0,1]\}$$

Approximate solution of problem (1) given in [3] by means of operator series is considered. It is shown that operator *G* is analytical according to Gatto at point $q=0$, if

$$M=\left\{q(x): q(x)\in L_2^+[0,1],\ \|q(x)\|_{L_2[0,1]}\leq 4\right\}$$

then on set *M* the series

$$G(x,\xi,q(.))=G(x,\xi,0)+\sum_{i=1}^{\infty}(-1)^i\int_0^1\ldots\int_0^1\left\{\prod_{j=1}^{i}G(z_{j-1},z_j,0)\,q(z_j)\right\}G(z_i,\xi,0)\,dz_1\ldots dz_i. \tag{3}$$

Where $z_0=x$,

$$G(x,\xi,0)=\begin{cases} x(1-\xi), 0\leq x\leq\xi, \\ \xi(1-x), \xi\leq x\leq 1, \end{cases} \tag{4}$$

are equally convergent, i.e. any $q(.)$ is aproximated by means of Green's function built for $q(.)=0$.

Consider operator interpolation approach to the solution of boundary value problem (1). In boundary value problem (1), change Green's function $G(x,\xi,q(.))$ by the *m*-order operator interpolation polynomial of the Newton type as follows:

$$G_m\left(x,\P,q(.)\right)=G\left(x,\P,0\right)+$$
$$\sum_{i=1}^{m}\int_0^1\cdots\int_0^1\left\{K_i\left(x,\P,z_1,\ldots,z_i\right)\prod_{j=1}^{i}H\left(z_j-z_{j-1}\right).\left[q\left(z_j\right)-h\left(j-1\right)\right]\right\}dz_1\cdots dz_i \tag{5}$$
$$z_0=0$$

For better illustration let us write formula (5) as follows

$$G_m\left(x,\P,q(.)\right)=G\left(x,\P,0\right)+\int_0^1\int_0^1 K_1\left(x,\P,z_1\right)q\left(z_1\right)dz_1+$$
$$\int_0^1\int_{z_1}^1 K_2\left(x,\P,z_1,z_2\right)q\left(z_1\right)\left[q\left(z_2\right)-h\right]dz_1dz_2+$$
$$\int_0^1\int_{z_1}^1\int_{z_2}^1 K_3\left(x,\P,z_1,z_2,z_3\right)q\left(z_1\right)\left[q\left(z_2\right)-h\right]\left[q\left(z_3\right)-2h\right]dz_1dz_2dz_3+\ldots+$$
$$\int_0^1\int_{z_1}^1\cdots\int_{z_{m-1}}^1 K_m\left(x,\P,z_1,z_2,\ldots,z_m\right)q\left(z_1\right)\left[q\left(z_2\right)-h\right]\ldots\left[q\left(z_m\right)-\left(m-1\right)h\right]dz_1dz_2\ldots dz_m \tag{6}$$

where the operator kernels

$$K_i\left(x,\P,z_1,\ldots,z_i\right)=\frac{(-1)^i}{h^i}\frac{\partial^i}{\partial z_1\ldots\partial z_i}G\left(x,\P,\text{¸}_i\right) \tag{7}$$
$$\text{¸}_i=h\sum_{j=1}^{i}H\left(.-z_j\right),\ i=1,2,\ldots,m,$$

Heavy-side function $H\left(z\right)=\begin{cases}1, & z>0,\\ 0, & z<0,\end{cases}$

Interpolation grid space $h=\dfrac{\left(c_m-c_0\right)}{\left(m+1\right)},\quad 0\le q\left(x\right)\le c,\quad c_0=\min\limits_{x\in[0,1]}q\left(x\right)$

$$c_m=\max_{x\in[0,1]}q\left(x\right),q\left(x\right)\in C[0,1].$$

Construct operators $K_1,K_2,\ldots$ kernels in formula (6) in two different ways: 1. Construct operator interpolation polynomial of the Newton type by the direct method (DM), where Green's auxiliary functions built by exact method will be used; 2. Construct the operator interpolation polynomial of the Newton type by the modified method (MM),where Green's auxiliary functions built by approximate method will be used.

Direct operator interpolation method (DOIM). To illustrate the direct method of construction of the operator kernels let us describe the process of construction of the operator kernel $K_1\left(x,\P,z_1\right)$ in detail. Operator kernel $K_1\left(x,\P,z_1\right)$ is given by

$$K_1\left(x,\P,z_1\right)=-\frac{1}{h}\frac{\partial}{\partial z_1}G\left(x,\P,\text{s}_1\right),\text{s}_1=hH\left(\cdot-z_1\right). \tag{8}$$

As is known [6, 7], Green's function $G\left(x,\P,\text{s}_1\right)$ satisfies the following boundary value problem

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} G\left(x,\P,\text{s}_1\right) - hH\left(x-z_1\right)G\left(x,\P,\text{s}_1\right) = -\text{¢}\left(x-\P\right), \\ G\left(0,\P,\text{s}_1\right) = G\left(1,\P,\text{s}_1\right) = 0, \quad 0 < x,\P < 1, \end{cases} \tag{9}$$

Where $\text{¢}\left(z\right)$ is Dirac delta function

$$\text{¢}\left(z\right) = \begin{cases} 0, z \neq 0, \\ \infty, z = 0, \end{cases} \int_{-\infty}^{+\infty} \text{¢}\left(z\right) dz = 1. \tag{10}$$

Differentiate system (9) by z_1 and multiply it by $-\dfrac{1}{h}$. As a result of elementary transformation for definition of $K_1\left(x,\P,z_1\right)$ we get boundary value problem

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} K_1\left(x,\P,z_1\right) - hH\left(x-z_1\right)K_1\left(x,\P,z_1\right) = \text{¢}\left(x-z_1\right)G\left(x,\P,\text{s}_1\right), \\ K_1\left(0,\P,z_1\right) = K_1\left(1,\P,z_1\right) = 0. \end{cases} \tag{11}$$

By the use of Green's function we obtain

$$K_1\left(x,\P,z_1\right) = \int_0^1 G\left(x,\text{¦},\text{s}_1\right)\text{¢}\left(\text{¦}-z_1\right)G\left(\text{¦},\P,\text{s}_1\right)d\text{¦} = -G\left(x,z_1,\text{s}_1\right)\cdot G\left(z_1,\P,\text{s}_1\right), \tag{12}$$

i.e. $K_1\left(x,\P,z_1\right)$ is represented as the product of two green functions $G\left(x,z_1,\text{s}_1\right)$ and $G\left(z_1,\P,\text{s}_1\right)$. These functions are not built directly. First the Green's function $G\left(x,\P,\text{s}_1\right)$ dependent on $\P$ and z_1 parameters is built, which will have two forms. And then the above mentioned Green's functions will follow from it as private cases. For construction of Green's function $G\left(x,\P,\text{s}_1\right)$ satisfying boundary value problem (9) its principal properties will be used. Function $G\left(x,\P,\text{s}_1\right)$ and its *x*-differential with *x* are continuous at point z_1, while the funtion itself is continuousat point $\P$ and its *x*-differential has a finite jump, which is equal to 1. For $\P \leq z_1$ the first form of Green's function is given by:

$$G^{1,1}\left(x,\P,\text{s}_1\right) = \begin{cases} c_1^{1,1}\left(\P,z_1\right) + c_2^{1,1}\left(\P,z_1\right)\cdot x, & x \in \left[0,\P\right], \\ c_3^{1,1}\left(\P,z_1\right) + c_4^{1,1}\left(\P,z_1\right)\cdot x, & x \in \left[\P,z_1\right], \\ c_5^{1,1}\left(\P,z_1\right)\cdot e^{-x\sqrt{h}} + c_6^{1,1}\left(\P,z_1\right)\cdot e^{x\sqrt{h}}, & x \in \left[z_1,1\right], \end{cases} \tag{13}$$

where

$$c_1^{1,1}\left(\P,z_1\right) \equiv 0$$

$$c_2^{1,1}\left(\P,z_1\right) = \frac{\left(\sqrt{h}\left(z_1-\P\right)+1\right)e^{(1-z_1)\sqrt{h}} + \left(\sqrt{h}\left(z_1-\P\right)-1\right)e^{-(1-z_1)\sqrt{h}}}{\left(z_1\sqrt{h}+1\right)e^{(1-z_1)\sqrt{h}} + \left(z_1\sqrt{h}-1\right)e^{-(1-z_1)\sqrt{h}}},$$

$$c_3^{1,1}\left(\P,z_1\right) = \P, c_4^{1,1}\left(\P,z_1\right) = c_2^{1,1}\left(\P,z_1\right) - 1, \tag{14}$$

$$c_5^{1,1}(\P, z_1) = \frac{e^{z_1\sqrt{h}}}{2\sqrt{h}}\left(\left(z_1\sqrt{h}-1\right)c_4^{1,1}(\P, z_1) - \P\sqrt{h}\right),$$

$$c_6^{1,1}(\P, z_1) = \frac{e^{-z_1\sqrt{h}}}{2\sqrt{h}}\left(\left(z_1\sqrt{h}+1\right)c_4^{1,1}(\P, z_1) + \P\sqrt{h}\right).$$

Consequently, Green's functions $G(x, z_1, \mathrm{s}_1)$ and $G(z_1, \P, \mathrm{s}_1)$ in formula (12) representing private cases of formula (13) will be given by:

$$G^{1,1}(z_1, \P, \mathrm{s}_1) = c_2^{1,1}(\P, z_1)\cdot z_1 + \P - z_1, \tag{15}$$

$$G^{1,1}(x, z_1, \mathrm{s}_1) = \begin{cases} c_2^{1,1}(z_1, z_1)\cdot x, & x\in[0, z_1], \\ c_5^{1,1}(z_1, z_1)\cdot e^{-x\sqrt{h}} + c_6^{1,1}(z_1, z_1)\cdot e^{x\sqrt{h}}, & x\in[z_1, 1]. \end{cases} \tag{16}$$

For $\P > z_1$ we have the second form of Green's function:

$$G^{2,1}(x, \P, \mathrm{s}_1) = \begin{cases} c_1^{2,1}(\P, z_1) + c_2^{2,1}(\P, z_1)\cdot x, & x\in[0, z_1], \\ c_3^{2,1}(\P, z_1)\cdot e^{-x\sqrt{h}} + c_4^{2,1}(\P, z_1)\cdot e^{x\sqrt{h}}, & x\in[z_1, \P], \\ c_5^{2,1}(\P, z_1)\cdot e^{-x\sqrt{h}} + c_6^{2,1}(\P, z_1)\cdot e^{x\sqrt{h}}, & x\in[\P, 1], \end{cases} \tag{17}$$

where

$$c_1^{2,1}(\P, z_1) \equiv 0,$$

$$c_2^{2,1}(\P, z_1) = \frac{e^{(1-\P)\sqrt{h}} - e^{-(1-\P)\sqrt{h}}}{\left(z_1\sqrt{h}+1\right)e^{(1-z_1)\sqrt{h}} + \left(z_1\sqrt{h}-1\right)e^{-(1-z_1)\sqrt{h}}},$$

$$c_3^{2,1}(\P, z_1) = \frac{z_1\sqrt{h}-1}{2\sqrt{h}}e^{z_1\sqrt{h}}\cdot c_2^{2,1}(\P, z_1),\ c_4^{2,1}(\P, z_1) = \frac{z_1\sqrt{h}+1}{2\sqrt{h}}e^{-z_1\sqrt{h}}\cdot c_2^{2,1}(\P, z_1), \tag{18}$$

$$c_5^{2,1}(\P, z_1) = c_3^{2,1}(\P, z_1) + \frac{1}{2\sqrt{h}}e^{\P\sqrt{h}}, \quad c_6^{2,1}(\P, z_1) = c_4^{2,1}(\P, z_1) - \frac{1}{2\sqrt{h}}e^{-\P\sqrt{h}}.$$

Similarly, Green's functions $G(x, z_1, \mathrm{s}_1)$ and $G(z_1, \P, \mathrm{s}_1)$ in formula (12) will be given by

$$G^{2,1}(z_1, \P, \mathrm{s}_1) = c_2^{2,1}(\P, z_1)\cdot z_1, \tag{19}$$

$$G^{2,1}(x, z_1, \mathrm{s}_1) = \begin{cases} c_2^{2,1}(z_1, z_1)\cdot x, & x\in[0, z_1], \\ c_5^{2,1}(z_1, z_1)\cdot e^{-x\sqrt{h}} + c_6^{2,1}(z_1, z_1)\cdot e^{x\sqrt{h}}, & x\in[z_1, 1]. \end{cases} \tag{20}$$

It is easy to check that

$$G^{1,1}(x, z_1, \mathrm{s}_1) \equiv G^{2,1}(x, z_1, \mathrm{s}_1),\ \ G^{1,1}(z_1, \P, \mathrm{s}_1) = G^{2,1}(z_1, \P, \mathrm{s}_1), \text{if } z_1 = \P .$$

The higher-order operator kernels can be built in the same way. The process of building of the higher-order operator kernels is a labor intensive process, but for the given boundary value problem the higher-order operator kernels are built once and for ever. The operator kernel dependent on the parameters $\P$, $z_1, z_2, \cdots, z_i$ is the sum of i! -number of items of the $i+1$ Green functions multiplication products. Note that when we use this method, with the increase of the approximating polynomials order all the kernels computed at the previous stage remain the same. Besides addition of new numbers the increase of approximating order causes just reduction of the grid values only.

Modified operator interpolation method (MOIM). The same way as in case of the direct method of building the operator kernels, let us describe the process of building the operator kernel $K_1(x,¶,z_1)$ in detail in order to illustrate the modified method. Decompose Green's function, which is the determining factor in determination of the operator interpolation polynomial kernels of Newton type, into the degrees of small parameters of *h*-degrees (*h*- the interpolation grid spacing).

Let us build the operator kernel $K_1(x,¶,z_1)$. Differentiate Green's function $G(x,¶,hH(\cdot - z_1))$ in *h*-degrees

$$G(x,¶,hH(\bullet - z_1)) = \sum_{j=0}^{\infty} G_j^{(1)}(x,¶,z_1)\,h^j \tag{21}$$

and introduce it into formula (8) defining the kernel K_1. The operator kernel $K_1(x,¶,z_1)$ will be given by:

$$K_1(x,¶,z_1) = -\frac{1}{h}\frac{\partial}{\partial z_1}\mathrm{G}(\mathrm{x},¶,hH(\bullet - z_1)) = -\sum_{i=0}^{\infty}\frac{\partial}{\partial z_1}G_j^{(1)}(x,¶,z_1)\,h^{j-1}. \tag{22}$$

As is known, Green's function satisfies the boundary value problem(9). Introduce the series of the Green's function differentiated with respect to *h* into formula (9) as polynomial. Take the problems similar to those used earlier (see (9)) to define $G_j^{(1)},\ j=0,1,2,\ldots$ Then, compute the expressions $\frac{\partial}{\partial z_1}G_j^{(1)},\ j=0,1,2,\ldots$.

Finally, we will have the following recurrent formulae:

$$G_{i+1}^{(1)}(x,¶,z_1) = -\int_{z_1}^{1} G(x,¦,0)\cdot G_i^{(1)}(¦,¶,z_1)\,d¦, \tag{23}$$

$$\frac{\partial}{\partial z_1}G_{i+1}^{(1)}(x,¶,z_1) = G(x,z_1,0)\cdot G_i^{(1)}(z_1,¶,z_1), \quad i=0,1,2,\ldots; \tag{24}$$

$$G_0^{(1)}(x,¶,z_1) = G(x,¶,0) = \begin{cases} x(1-¶), & 0 \le x \le ¶, \\ ¶(1-x), & ¶ \le x \le 1, \end{cases} \tag{25}$$

$$\frac{\partial}{\partial z_1}G_0^{(1)}(x,¶,z_1) = 0. \tag{26}$$

$$G_0^{(1)}(x,¶,z_1) = G(x,¶,0) = \begin{cases} x(1-¶), & 0 \le x \le ¶, \\ ¶(1-x), & ¶ \le x \le 1, \end{cases}$$

Let us write some coefficients in detail:

$$G_1^{(1)}(x,¶,z_1) = -\int_{z_1}^{1} G(x,¦,0)*G(¦,¶,0)\,d¦ =$$

$$= -\begin{cases} (1-x)(1-¶)\left(\frac{¶^3}{3}-\frac{z_1^3}{3}\right)+¶(1-x)\left(\frac{x^2}{2}-\frac{x^3}{3}-\frac{¶^2}{2}+\frac{¶^3}{3}\right)+x¶\left(\frac{1}{3}-x+x^2-\frac{x^3}{3}\right), & ¶ \le x, \\ (1-x)(1-¶)\left(\frac{x^3}{3}-\frac{z_1^3}{3}\right)+x(1-¶)\left(\frac{¶^2}{2}-\frac{¶^3}{3}-\frac{x^2}{2}+\frac{x^3}{3}\right)+x¶\left(\frac{1}{3}-¶+¶^2-\frac{¶^3}{3}\right), & ¶ \ge x, \end{cases} \tag{27}$$

$$G_2^{(1)}\left(x,\P,z_1\right)=$$

$$=\begin{cases}\left(1-x\right)\left(1-\P\right)\left[-\frac{1}{30}\P^5-\frac{1}{3}z_1^3\left(\frac{1}{2}\P^2-\frac{1}{3}\P^3\right)+\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)\left(\frac{1}{3}\P^3-\frac{1}{3}z_1^3\right)-\frac{1}{9}z_1^6+\frac{1}{5}z_1^5\right]+\\ \left(1-x\right)\left(1-\P\right)\left[-\frac{1}{30}x^5-\frac{1}{3}z_1^3\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)+\left(\frac{1}{2}\P^2-\frac{1}{3}\P^3\right)\left(\frac{1}{3}x^3-\frac{1}{3}z_1^3\right)-\frac{1}{9}z_1^6+\frac{1}{5}z_1^5\right]+\end{cases}$$

$$\begin{cases}+\left(1-x\right)\P*[-\frac{1}{30}\P^5+\frac{1}{24}\P^4+\frac{1}{30}x^5-\frac{1}{8}x^4+\frac{1}{9}x^3+\frac{1}{3}z_1^3*\left(\frac{1}{3}\P^3-\P^2+\P-\frac{1}{3}\right)-\\ +\left(1-\P\right)x*[-\frac{1}{30}x^5+\frac{1}{24}x^4+\frac{1}{30}\P^5-\frac{1}{8}\P^4+\frac{1}{9}\P^3+\frac{1}{3}z_1^3*\left(\frac{1}{3}x^3-x^2+x-\frac{1}{3}\right)-\end{cases}$$

$$\begin{cases}-\left(\frac{1}{2}\P^2-\frac{1}{3}\P^3\right)*\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)]+x\left(1-\P\right)*\frac{1}{9}*\left(1-3x+3x^2-x^3\right)\left(\P^3-z_1^3\right)+\\ -\left(\frac{1}{2}\P^2-\frac{1}{3}\P^3\right)*\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)]+\P\left(1-x\right)*\frac{1}{9}*\left(1-3\P+3\P^2-\P^3\right)\left(x^3-z_1^3\right)+\end{cases}$$

$$\begin{cases}+x*\P*\left[\frac{1}{45}+\frac{1}{30}x^5-\frac{1}{6}x^4+\frac{5}{18}x^3-\frac{1}{6}x^2+\left(\frac{1}{2}\P^2-\frac{1}{3}\P^3\right)*\left(\frac{1}{3}x^3-x^2+x-\frac{1}{3}\right)\right], & \P\le x,\\ +x*\P*\left[\frac{1}{45}+\frac{1}{30}\P^5-\frac{1}{6}\P^4+\frac{5}{18}\P^3-\frac{1}{6}\P^2+\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)*\left(\frac{1}{3}\P^3-\P^2+\P-\frac{1}{3}\right)\right], & \P\ge x.\end{cases} \tag{28}$$

Theoretically, as in case of interpolation approach, computation of the higher-order operator kernels is not difficult, but the volume of work substantially increases (i.e. technical difficulties arise).

Approximate solution of the boundary value problem in the modified approach will be the same as in case of direct method. However, this approach is somewhat improved compared to the previous one. Operator kernels $G\left(x,\P,0\right)$ can be constructed by the use of Green's function only, though neither this approach can avoid the difficulty of multiple calculations of integrals.

The algorithms of approximate solution of boundary value problem. For approximate solution of boundary value problem (1), apply operator interpolation polynomials built by means of two different methods (direct and modified).

Let us construct several approximate solutions of the boundary value problem (1) in detail. In *m*-approximation the $u_m\left(x\right)$ solution is given by

$$u_m\left(x\right)=\int_0^1 G_m\left(x,\P,q(.)\right)f\left(\P\right)d\P,$$

Where $G_m\left(x,\P,q(.)\right)$ is defined by formula (5) because

$$G_0\left(x,\P,q(.)\right)=G\left(x,\P,0\right)=\begin{cases}x\left(1-\P\right), & 0\le x\le\P,\\ \P\left(1-x\right), & \P\le x\le 1,\end{cases} \tag{29}$$

Therefore, for computation of the zero-approximation we get

$$u_0(x) = \int_0^1 G(x, \P, 0)\, f(\P)\, d\P = \int_0^x \P\,(1-x)\, f(\P)\, d\P + \int_x^1 x(1-\P)\, f(\P)\, d\P =$$

$$= (1-x)\int_0^x \P\, f(\P)\, d\P + x\int_x^1 (1-\P)\, f(\P)\, d\P, \qquad (30)$$

i.e., for computation of zero-approximation (for *m*= 0) it is necessary to compute the sum of two single integrals.

The first approximation (i.e.for *m* = 1) is given by

$$u_1(x) = \int_0^1 G_1(x, \P, q(.))\, f(\P)\, d\P =$$

$$= \int_0^1 \left[G(x, \P, 0) + \int_0^1 K_1(x, \P, z_1)\, q(z_1)\, dz_1 \right] f(\P)\, d\P =$$

$$\int_0^1 G(x, \P, 0)\, f(\P)\, d\P + \int_0^1 \left[\int_0^1 K_1(x, \P, z_1)\, q(z_1)\, dz_1 \right] f(\P)\, d\P = u_0(x) - v_1(x), \qquad (31)$$

where

$$v_1(x) = -\int_0^1 \left[\int_0^1 K_1(x, \P, z_1)\, q(z_1)\, dz_1 \right] f(\P)\, d\P =$$

$$= \int_0^1 \left[\int_0^1 G(x, z_1, \text{¸}_1)\, G(z_1, \P, \text{¸}_1)\, q(z_1)\, dz_1 \right] f(\P)\, d\P. \qquad (32)$$

Kernel $K_1(x, \P, z_1)$ is defined from formula (12). We can stop here and describe the lower integral function in a complex way. But for reliability of the computation results it is better to consider the division of the integral special cases. Here, for different relation between the parameters x, ¶, z_1 we take into consideration different types of Green's function (15, 16, 19, 20). Since $v_1(x)$ can be given by

$$v_1(x) = \int_0^x \int_0^{\P} \ldots dz_1 d\P + \int_0^x \int_{\P}^x \ldots dz_1 d\P + \int_0^x \int_x^1 \ldots dz_1 d\P + \int_x^1 \int_0^x \ldots dz_1 d\P +$$

$$+ \int_x^1 \int_x^{\P} \ldots dz_1 d\P + \int_x^1 \int_{\P}^1 \ldots dz_1 d\P, \qquad (33)$$

i.e. for the first approximation (*m*=1) we have the sum of 6 double integrals.

In general case, for computation of $v_i(x)$ we need to find the sum of $(i+1)$-times integral $(i+1)(i+2)$. As noted above, with the increase of *m* the workload is substantially increasing. In the formulae constructed functions $f(x)$ and $q(x)$ participate as parameters.

Table 1. Test problem 1. Variable coefficient $q\left(x\right)=1+x^2$, **the right hand-side** $f\left(x\right)=-\left(x^4-x^3+x^2-x-2\right)$**, exact solution** $u\left(x\right)=x\left(1-x\right)$**.**

x/u	u_{Exact}	*Methods*	$u_0(x)$	*Error*	$u_1(x)$	*Error*	$u_2(x)$	*Error*
0.25	0.1875	DM	0.21018	0.02268	0.16485	0.02265	0.16796	0.01954
		MM	0.21018	0.02268	0.17934	0.00816	0.18130	0.00621
0.50	0.25	DM	0.28333	0.03333	0.22146	0.02854	0.22675	0.02325
		MM	0.28333	0.03333	0.24351	0.00649	0.24712	0.00288
0.75	0.1875	DM	0.21262	0.02512	0.17190	0.01560	0.17644	0.01106
		MM	0.21262	0.02512	0.17768	0.00982	0.18174	0.00576

Table 2. Test problem 2. Variable coefficient $q\left(x\right)=1+x^2$, **the right hand-side** $f\left(x\right)=-\left(x^5-x^4+x^3-x^2-3.75x+075\right)/\sqrt{x}$, **exact solution** $u\left(x\right)=x\sqrt{x}\left(1-x\right)$.

x/u	u_{Exact}	*Methods*	$u_0(x)$	*Error*	$u_1(x)$	*Error*	$u_2(x)$	*Error*
0.25	0.09375	DM	0.10877	0.01502	0.10849	0.01474	0.11093	0.01718
		MM	0.10877	0.01502	0.08600	0.00775	0.08730	0.00645
0.50	0.17678	DM	0.20051	0.02373	0.17939	0.00262	0.18334	0.00657
		MM	0.20051	0.02373	0.17317	0.00361	0.17649	0.00028
0.75	0.16238	DM	0.18145	0.01907	0.16052	0.00186	0.16400	0.00162
		MM	0.18145	0.01907	0.16037	0.00201	0.16396	0.00158

Table 3. Test problem 3. Variable coefficient $q\left(x\right)=1+x^2$, **the right hand-side** $f\left(x\right)=\left(1+®^2+x^2\right)\sin ® x$ **exact solution** $u\left(x\right)=\sin ® x$**.**

x/u	u_{Exact}	*Methods*	$u_0(x)$	*Error*	$u_1(x)$	*Error*	$u_2(x)$	*Error*
0.25	0.70711	DM	0.79473	0.08762	0.66236	0.04475	0.67183	0.03528
		MM	0.79473	0.08762	0.69647	0.01064	0.70456	0.00254
0.50	1.00000	DM	1.12956	0.12956	0.89629	0.10371	0.91486	0.08514
		MM	1.12956	0.12956	0.95838	0.04162	0.97554	0.02446
0.75	0.70711	DM	0.80384	0.09673	0.65948	0.04762	0.67017	0.03693
		MM	0.80384	0.09673	0.67535	0.03176	0.68856	0.01854

Note: The pre-integral function can be discribed as a finite function transforming multiple integrals into repetitive ones.Then the approximate values can be computed by the quadratic formula.

Numerical experiments. For approximate solution of the boundary value problem a software product is worked out in the software system MATLAB and is realized on a PC. For some practical tasks it is often sufficient to take no more than a second order operator polynomials. Numerical results of the zero, first and second approximations of solution are obtained for different test problems.

For illustration let us consider some test problems. Both in direct and modified methods we take the second order for the integration accuracy $£ = 0.001$ in decomposition of Green's function into *h*- degrees, which conditions $O(h^2)$ accuracy in the modified method.

Conclusions. The operator interpolation method is a numerical-analytical method. The mechanism of functional analysis is used. Often for approximate solution of some practical problem it is sufficient to take no more than a second-order operator interpolation polynomial. Using operator-interpolation methods with the increase of the order of approximating polynomials the kernels computed at the previous stage remain the same. Nevertheless, construction of the operator kernels of higher order is a labour consuming process. Therefore, technically it is difficult to achieve great precision.

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ა. პაპუკაშვილი

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(წარმოდგენილია აკადემიის წევრის ე.ნადარაიას მიერ)

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