საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 10, №3, 2016
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 10, no. 3, 2016

Mathematics

Stochastic Integral Representation of One Nonsmooth Brownian Functional

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ABSTRACT. We developed one method of obtaining the stochastic integral representation of nonsmooth (in Malliavin sense) Brownian functional and found explicit form of integrands in this representation. Because the Clark-Ocone's well-known method here is not applicable, we try to obtain the Clark integral representation with known integrand applying a nonconventional method. We consider a case, when functional represents the Lebesgue integral (with respect to time variable) from certain stochastically nonsmooth square integrable process which is also not smooth functional. It turned out that the requirement of smoothness of functional can be weakened by the requirement of smoothness only of its conditional mathematical expectation. Despite the fact that integrand of the functional considered by us satisfies the last requirement, its average (with respect to *dt*) functional has not the same property. At first, we give the stochastic integral representation for integrand of our functional and then due to the stochastic type Fubiny theorem we obtain the desired integral representation. *© 2016 Bull. Georg. Natl. Acad. Sci.*

Key words: Malliavin derivative, Brownian functional, Clark-Ocone representation, Fubiny theorem

Introduction and Auxiliary Results

As it is well-known from Ito's calculus, stochastic integral from square integrable adapted process is square integrable martingale. The answer on the inverse question: is it possible to represent the square integrable martingale adapted to the natural filtration of Brownian motion, as the stochastic integral is given by well-known Clark formula [1]. In particular, let B_t ($t \in [0,T]$) be standard Brownian motion and $\Im_t$ is a natural filtration generated by this Brownian motion. If F is a square integrable $\Im_T$-measurable random variable, then there exists square integrable $\Im_t$-adapted random process φ_t such that $F = EF + \int_0^T \varphi_t dB_t$. On the other hand, finding of explicit expression for φ_t is a very difficult problem. In this direction, one general result called Clark-Ocone formula is known [2], according to which $\varphi_t = E(D_t F \mid \Im_t)$, where D_t is the so-called

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Malliavin stochastic derivative. But, on the one hand, here the stochastically smoothness is required and, on the other hand, even in case of smoothness, calculations of Malliavin derivative and conditional mathematical expectation are rather difficult.

The next step in this direction was taken by Ma, Protter and Martin [3], they offered the concept of stochastic derivative and generalized stochastic integral for so-called normal martingales class and generalized Clark's formula for functionals from the class $D_{2,1}^{M}$ (the functional $F=\sum_{n=0}^{\infty} I_n(f_n)$ belongs to the space $D_{2,1}^{M}$ if and only if $\sum_{n=1}^{\infty} nn!\| f_n \|_{L_2([0,T]^n)}^2 < \infty$). We [4] introduced the space $D_{p,1}^{M}$, $1<p<2$ ($D_{p,1}^{M}$ the Banach space which is the closure of $D_{2,1}^{M}$ under the following norm $\| F \|_{p,1} := E(\| F \|_{L_p} + \| D.F \|_{L_2([0,T])})$ and extended the Ocone-Haussmann-Clark formula for functionals from this space. Absolutely different method for finding φ_t was offered by Shyriaev, Yor and Graversen [5, 6], which was based on using of Ito's (generalized) formula and Levy's theorem for associated to F Levy's martingale $m_t = E(F \mid \Im_t)$. We [7] introduced the new construction of stochastic derivative of Poisson functional and established the explicit expression for the integrand of Clark representation.

In all the cases described above F was stochastically smooth. We [8] considered the case when F is not stochastically smooth, but from associated with F Levy's martingale one can select a stochastically smooth subsequence and in this case we gave the method for finding the integrand. In particular, we generalized the Clark-Ocone formula in case, when functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable and established the method finding this integrand.

It is well-known that if random variable is stochastically differentiable in Malliavin sense, then its conditional mathematical expectation is differentiable too [5]. In particular, if $F \in D_{2,1}$, then $E(F \mid \Im_s) \in D_{2,1}$ (where $D_{2,1} := D_{2,1}^{B}$) and $D_t[E(F \mid \Im_s)] = E(D_t F \mid \Im_s) I_{[0,s]}(t)$.

On the other hand, it is possible that conditional expectation can be smooth even if random variable is not stochastically smooth [8]. For example, it is well-known that $I_{\{B_t \le x\}} \notin D_{2,1}$ (indicator of event A is Malliavin differentiable if and only if probability $P(A)$ is equal to zero or one [9]), but for all $s \in [0,T)$: $E[I_{\{B_T \le x\}} \mid \Im_s] = \Phi((x-B_s)/\sqrt{T-s}) \in D_{2,1}$.

Here we investigate a different case, when functional represents the Lebesgue integral from stochastically nonsmooth square integrable process with respect to time variable. In particular, we consider the functional of integral type $G = \int_0^T B_t I_{\{a \le S_t \le b\}} dt$ (where S_t is the geometrical Brownian motion corresponding to B_t). According to the Theorem 2 from [10] this functional does not belong to $D_{2,1}$. Moreover, the conditional mathematical expectation of this functional is not stochastically smooth, because we have:

$$E(G \mid \Im_s) = \int_0^s B_t I_{\{a \le S_t \le b\}} dt + \int_s^T E[B_t I_{\{a \le S_t \le b\}} \mid \Im_s] dt,$$

where the first summand is not differentiable, but the second summand is differentiable in Malliavin sense (the average, with respect to dt, functional from stochastically smooth processes is also stochastically smooth). Therefore, here neither the known method of Clark-Ocone is applicable nor our method which generalizes it [8, 10- 12].

For convenience of statement we will give some auxiliary results below.

Let $p(u,t,B_u,A)$ be the transition probability of the Brownian motion B, i.e. $P[B_t \in A \mid \Im_u] = p(u,t,B_u,A)$, where $0 \le u \le t$, A is a Borel subset of R and

$$p(u,t,x,A) = \frac{1}{\sqrt{2\pi(t-u)}} \int_A \exp\{-\frac{(y-x)^2}{2(t-u)}\} dy .$$

For computation of conditional mathematical expectation below we use the well-known statement:

Proposition 1. For all bounded or positive measurable function *f* we have the following relation

$$E[f(B_t) \mid \Im_u] = \int_{-\infty}^{\infty} f(y) p(u,t,B_u,dy) \qquad (P\text{-a.s.}).$$

Proposition 2. Let $\psi : R^m \to R$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F^1,...,F^2)$ is a random vector whose components belong to the space $D_{2,1}$. Then $\psi(F) \in D_{2,1}$, and

$$D_t(\psi(F)) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \psi(F) D_t F^i$$

(see, [9, Proposition 1.2.3.]).

Theorem 1. *Suppose that* $g_t = E[F \mid \Im_t]$ *is Malliavin differentiable (*$g_t(\cdot) \in D_{2,1}$*) for almost all* $t \in [0,T)$*. Then we have the following stochastic integral representation*

$$g_T = F = EF + \int_0^T \nu_u dB_u \qquad (P\text{-a.s.}),$$

where

$$\nu_u := \lim_{t \uparrow T} E[D_u g_t \mid \Im_u] \quad \text{in the } L_2([0,T] \times \Omega)$$

(see, [8, theorem from section 2]).

Stochastic Integral Representation Theorem

Let on the complete probability space $(\Omega, \Im, P)$ be given the Brownian motion $B = (B_t)$, $t \in [0,T]$ and $(\Im_t)$, $t \in [0,T]$ be the natural filtration generated by the Brownian motion B. Consider the geometrical Brownian motion described by the equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = 1$$

(where $\mu \in R$ is appreciation rate and $\sigma > 0$ is volatility coefficient) or

$$S_t = \exp\{\sigma B_t + (\mu - \sigma^2/2)t\} .$$

Theorem 2. *For any real number c>0 and* $\tau \in (0,T]$ *the random variable* $B_\tau I_{\{S_\tau \le c\}}$ *have the following stochastic integral representation*

$$B_\tau I_{\{S_\tau \le c\}} = -\sqrt{\tau}\varphi(\frac{\ln c - r\tau}{\sigma\sqrt{\tau}}) +$$

$$+\int_0^\tau [\Phi(\frac{\ln c - r\tau - \sigma B_u}{\sigma\sqrt{\tau-u}}) - \frac{\ln c - r\tau}{\sigma\sqrt{\tau-u}} \varphi(\frac{\ln c - r\tau - \sigma B_u}{\sigma\sqrt{\tau-u}})] dB_u , \qquad (1)$$

where $r=\mu-\sigma^2/2$ (here and below $\Phi_{\upsilon,\lambda}$ is the normal distribution function with parameters υ and λ and $\varphi_{\upsilon,\lambda}$ is its density function; $\Phi:=\Phi_{0,1}$ and $\varphi:=\varphi_{0,1}$, i. e.

$$\varphi_{\upsilon,\lambda}(x)=\frac{1}{\sqrt{2\pi\lambda}}\exp\{-\frac{(x-\upsilon)^2}{2\lambda}\} \quad \text{and} \quad \Phi_{\upsilon,\lambda}(x)=\frac{1}{\sqrt{2\pi\lambda}}\int_{-\infty}^{x}\exp\{-\frac{(y-\upsilon)^2}{2\lambda}\}dy).$$

Proof. According to the Proposition 1, using the standard technique of integration and the well-known property of the normal distribution and its density functions, it is not difficult to see that

$$g_t^\tau := E[B_\tau I_{\{S_\tau\le c\}}\mid\Im_t]=E[B_\tau I_{\{B_\tau\le(\ln c-r\tau)/\sigma\}}\mid\Im_t]=$$

$$=\frac{1}{\sqrt{2\pi(\tau-t)}}\int_{-\infty}^{\infty}xI_{\{x\le(\ln c-r\tau)/\sigma\}}\exp\{-\frac{(x-B_t)^2}{2(\tau-t)}\}dx=$$

$$=\frac{1}{\sqrt{2\pi(\tau-t)}}\int_{-\infty}^{\infty}(x-B_t)I_{\{x\le(\ln c-r\tau)/\sigma\}}\exp\{-\frac{(x-B_t)^2}{2(\tau-t)}\}dx+$$

$$+\frac{B_t}{\sqrt{2\pi(\tau-t)}}\int_{-\infty}^{\infty}I_{\{x\le(\ln c-r\tau)/\sigma\}}\exp\{-\frac{(x-B_t)^2}{2(\tau-t)}\}dx=$$

$$=-\frac{\tau-t}{\sqrt{2\pi(\tau-t)}}\int_{-\infty}^{\infty}I_{\{x\le(\ln c-r\tau)/\sigma\}}d(\exp\{-\frac{(x-B_t)^2}{2(\tau-t)}\})+$$

$$+B_t\Phi_{B_t,\tau-t}(\frac{\ln c-r\tau}{\sigma})=$$

$$=-\sqrt{\tau-t}\,\varphi(\frac{\ln c-r\tau-\sigma B_t}{\sigma\sqrt{\tau-t}})+B_t\Phi(\frac{\ln c-r\tau-\sigma B_t}{\sigma\sqrt{\tau-t}})\,. \tag{2}$$

Therefore, according to the Proposition 2, the random variable $g_t^\tau=E[B_\tau I_{\{S_\tau\le c\}}\mid\Im_t]$ is Malliavin differentiable $g_t^\tau\in D_{2,1}$ for all $t\in[0,\tau)$. Hence, due to the Theorem 1, we have the following stochastic integral representation:

$$B_\tau I_{\{S_\tau\le c\}}=E[B_\tau I_{\{S_\tau\le c\}}]+\int_0^\tau \nu_u^\tau dB_u \qquad (P\text{-a.s.}), \tag{3}$$

where

$$\nu_u^\tau:=\lim_{t\uparrow\tau}E[D_u g_t^\tau\mid\Im_u] \quad \text{in the} \quad L_2([0,T]\times\Omega)\,. \tag{4}$$

It is not difficult to see that

$$E[B_\tau I_{\{S_\tau\le c\}}]=\int_{-\infty}^{\infty}xI_{\{x\le(\ln c-r\tau)/\sigma\}}\varphi_{0,\tau}(x)dx=$$

$$=-\tau\frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}I_{\{x\le(\ln c-r\tau)/\sigma\}}d(\exp\{-\frac{x^2}{2\tau}\})=-\sqrt{\tau}\,\varphi(\frac{\ln c-r\tau}{\sigma\sqrt{\tau}})\,. \tag{5}$$

On the other hand, using the Proposition 2, from the relation (2) we can write

$$D_u g_t^\tau = \varphi(\frac{\ln c - r\tau - \sigma B_t}{\sigma\sqrt{\tau - t}}) I_{[0,t]}(u) +$$

$$+\Phi(\frac{\ln c - r\tau - \sigma B_t}{\sigma\sqrt{\tau - t}}) I_{[0,t]}(u) - \frac{B_t}{\sqrt{\tau - t}} \varphi(\frac{\ln c - r\tau - \sigma B_t}{\sigma\sqrt{\tau - t}}) I_{[0,t]}(u) :=$$

$$:= J_1(t,\tau,B_t) I_{[0,t]}(u) + J_2(t,\tau,B_t) I_{[0,t]}(u) - \frac{1}{\sqrt{\tau - t}} J_3(t,\tau,B_t) I_{[0,t]}(u) . \quad (6)$$

Due to the Proposition 1, using again the standard technique of integration and the well-known property of the normal distribution density function, we easily ascertain that

$$E[J_1(t,\tau,B_t) \mid \Im_u] := E[\varphi(\frac{\ln c - r\tau - \sigma B_t}{\sigma\sqrt{\tau - t}}) \mid \Im_u] =$$

$$= \frac{1}{\sqrt{2\pi(t-u)}} \int_{-\infty}^{\infty} \varphi(\frac{\ln c - r\tau - \sigma x}{\sigma\sqrt{\tau - t}}) \exp\{-\frac{(x - B_u)}{2(t-u)}\} dx =$$

$$= \frac{1}{2\pi\sqrt{(t-u)}} \exp\{-\frac{(\ln c - r\tau - \sigma B_u)^2}{2\sigma^2(\tau - u)}\} \times$$

$$\times \int_{-\infty}^{\infty} \exp\{-\frac{\left[x - \frac{(\ln c - r\tau)(t-u) + \sigma B_u(\tau - t)}{\sigma(\tau - u)}\right]^2}{2\frac{(\tau - t)(t-u)}{\tau - u}}\} dx =$$

$$= \frac{1}{2\pi\sqrt{(t-u)}} \exp\{-\frac{(\ln c - r\tau - \sigma B_u)^2}{2\sigma^2(\tau - u)}\} \sqrt{2\pi\frac{(\tau - t)(t-u)}{\tau - u}} =$$

$$= \sqrt{\frac{\tau - t}{2\pi(\tau - u)}} \exp\left\{-\frac{(\ln c - r\tau - \sigma B_u)^2}{2\sigma^2(\tau - u)}\right\}.$$

Hence, it is clear that

$$\lim_{t \uparrow \tau} E[J_1(t,\tau,B_t) I_{[0,t]}(u) \mid \Im_u] = 0 . \quad (7)$$

Farther, using the Proposition 1, we obtain that

$$E[J_2(t,\tau,B_t) \mid \Im_u] := E[\Phi(\frac{\ln c - r\tau - \sigma B_t}{\sigma\sqrt{\tau - t}}) \mid \Im_u] =$$

$$= \frac{1}{\sqrt{2\pi(t-u)}} \int_{-\infty}^{\infty} \Phi(\frac{\ln c - r\tau - \sigma x}{\sigma\sqrt{\tau - t}}) \exp\{-\frac{(x - B_u)}{2(t-u)}\} dx .$$

Therefore, due to the relation

$$\lim_{t \uparrow \tau} \Phi(\frac{y}{\sqrt{\tau - t}}) = \begin{cases} 0, & y < 0; \\ 0.5, & y = 0; \\ 1, & y > 0, \end{cases}$$

using the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{t\uparrow\tau} E[J_2(t,\tau,B_t)I_{[0,t]}(u)\,|\,\Im_u]=$$

$$=\frac{1}{\sqrt{2\pi(\tau-u)}}\int_{-\infty}^{\infty} I_{\{\ln c-r\tau-\sigma x>0\}}\exp\{-\frac{(x-B_u)}{2(\tau-u)}\}dxI_{[0,\tau]}(u)=$$

$$=\Phi_{B_u,\tau-u}(\frac{\ln c-r\tau}{\sigma})I_{[0,\tau]}(u)=\Phi(\frac{\ln c-r\tau-\sigma B_u}{\sigma\sqrt{\tau-u}})I_{[0,\tau]}(u). \tag{8}$$

At last, by analogy of the transformations made at calculation of the conditional mathematical expectation $E[J_1(t,\tau,B_t)\,|\,\Im_u]$, using the integration by parts formula, it is not difficult to see that

$$E[J_3(t,\tau,B_t)\,|\,\Im_u]:=E[B_t\varphi(\frac{\ln c-r\tau-\sigma B_t}{\sigma\sqrt{\tau-t}})\,|\,\Im_u]=$$

$$=\frac{1}{\sqrt{2\pi(t-u)}}\int_{-\infty}^{\infty} x\varphi(\frac{\ln c-r\tau-\sigma x}{\sigma\sqrt{\tau-t}})\exp\{-\frac{(x-B_u)}{2(t-u)}\}dx=$$

$$=\frac{1}{2\pi\sqrt{(t-u)}}\exp\{-h_1\}\int_{-\infty}^{\infty} x\exp\{-\frac{(x-h_2)}{2h_3}\}dx=$$

$$=\frac{h_3}{2\pi\sqrt{(t-u)}}\exp\{-h_1\}\int_{-\infty}^{\infty} d(\exp\{-\frac{(x-h_2)}{2h_3}\})+$$

$$+\frac{h_2}{2\pi\sqrt{(t-u)}}\exp\{-h_1\}\int_{-\infty}^{\infty}\exp\{-\frac{(x-h_2)}{2h_3}\}dx,$$

where

$$h_1:=h_1(\tau,u,B_u)=\frac{(\ln c-r\tau-\sigma B_u)^2}{2\sigma^2(\tau-u)},$$

$$h_2:=h_2(\tau,t,u,B_u)=\frac{(\ln c-r\tau)(t-u)+\sigma B_u(\tau-t)}{\sigma(\tau-u)},$$

$$h_3:=h_3(\tau,t,u)=\frac{(\tau-t)(t-u)}{\tau-u}.$$

Hence, according to the well-known property of the normal distribution density function, we can conclude that

$$E[J_3(t,\tau,B_t)\,|\,\Im_u]=-\frac{h_3}{2\pi\sqrt{(t-u)}}\exp\{-h_1\}\exp\{-\frac{(x-h_2)}{2h_3}\}\Big|_{x=-\infty}^{\infty}+$$

$$+\frac{h_2}{2\pi\sqrt{t-u}}\exp\{-h_1\}\sqrt{2\pi}\sqrt{h_3}=\frac{h_2\sqrt{h_3}}{\sqrt{2\pi(t-u)}}\exp\{-h_1\}.$$

Therefore, we obtain that

$$\lim_{t\uparrow\tau} E[\frac{1}{\sqrt{\tau-t}}J_3(t,\tau,B_t)I_{[0,t]}(u)\,|\,\Im_u]=\exp\{-h_1\}\times$$

$$\times \lim_{t \uparrow \tau}\{\frac{(\ln c - r\tau)(t-u)+\sigma B_u(\tau - t)}{\sqrt{2\pi}\sigma(\tau-u)^{3/2}} I_{[0,t](u)}\} =$$

$$= \exp\{-h_1\} \times \frac{\ln c - r\tau}{\sigma\sqrt{2\pi(\tau-u)}} I_{[0,\tau](u)} = \frac{\ln c - r\tau}{\sigma\sqrt{\tau-u}}\varphi(\frac{\ln c - r\tau - \sigma B_u}{\sigma\sqrt{\tau-u}}) I_{[0,\tau](u)} . \tag{9}$$

Combining now the relations (7), (8) and (9), we easily ascertain that

$$v_u^\tau = [\Phi(\frac{\ln c - r\tau - \sigma B_u}{\sigma\sqrt{\tau-u}}) - \frac{\ln c - r\tau}{\sigma\sqrt{\tau-u}}\varphi(\frac{\ln c - r\tau - \sigma B_u}{\sigma\sqrt{\tau-u}})] I_{[0,\tau]}(u), \tag{10}$$

which with the relations (3), (4) and (5) complete the proof of theorem.

Corollary 1. *In the case* $\mu = \sigma^2 / 2$ *for any real number* $c > 0$ *and* $\tau \in (0,T]$ *the random variable* $B_\tau I_{\{S_\tau \le c\}}$ *admits the following stochastic integral representation*

$$B_\tau I_{\{S_\tau \le c\}} = -\sqrt{\tau}\varphi(\frac{\ln c}{\sigma\sqrt{\tau}}) +$$

$$+\int_0^\tau [\Phi(\frac{\ln c - \sigma B_u}{\sigma\sqrt{\tau-u}}) - \frac{\ln c}{\sigma\sqrt{\tau-u}}\varphi(\frac{\ln c - \sigma B_u}{\sigma\sqrt{\tau-u}})] dB_u . \tag{11}$$

Corollary 2. *For any real number c and* $\tau \in (0,T]$ *the random variable* $B_\tau I_{\{B_\tau \le c\}}$ *has the following stochastic integral representation*

$$B_\tau I_{\{B_\tau \le c\}} = -\sqrt{\tau}\varphi(\frac{c}{\sqrt{\tau}}) + \int_0^\tau [\Phi(\frac{c - B_u}{\sqrt{\tau-u}}) - \frac{c}{\sqrt{\tau-u}}\varphi(\frac{c - B_u}{\sqrt{\tau-u}})] dB_u . \tag{12}$$

Theorem 3. *For any real positive numbers* $a < b$ *the integral type functional* $G = \int_0^T B_t I_{\{a \le S_t \le b\}} dt$ *admits the following stochastic integral representation*

$$\int_0^T B_t I_{\{a \le S_t \le b\}} dt = -\int_0^T [\sqrt{t}\varphi(\frac{\ln c - rt}{\sigma\sqrt{t}})] |_{c=a}^b dt +$$

$$+\int_0^T \{\int_u^T [\Phi(\frac{\ln c - rt - \sigma B_u}{\sigma\sqrt{t-u}}) - \frac{\ln c - rt}{\sigma\sqrt{t-u}}\varphi(\frac{\ln c - rt - \sigma B_u}{\sigma\sqrt{t-u}})] |_{c=a}^b dt\} dB_u . \tag{13}$$

Proof. Taking from the both parts of expression (1) integral with respect to $d\tau$, using the stochastic type Fubiny theorem (see, [13, Corollary of the Lemma IV.2.4]), it is not difficult to see that the following stochastic integral representation is fulfilled

$$\int_0^T B_\tau I_{\{S_\tau \le c\}} d\tau = -\int_0^T \sqrt{\tau}\varphi(\frac{\ln c - r\tau}{\sigma\sqrt{\tau}}) d\tau +$$

$$+\int_0^T \int_0^\tau [\Phi(\frac{\ln c - r\tau - \sigma B_u}{\sigma\sqrt{\tau-u}}) - \frac{\ln c - r\tau}{\sigma\sqrt{\tau-u}}\varphi(\frac{\ln c - r\tau - \sigma B_u}{\sigma\sqrt{\tau-u}})] dB_u d\tau =$$

$$= -\int_0^T \sqrt{\tau}\varphi(\frac{\ln c - r\tau}{\sigma\sqrt{\tau}}) d\tau +$$

$$+\int_0^T \{\int_u^T [\Phi(\frac{\ln c - r\tau - \sigma B_u}{\sigma\sqrt{\tau-u}}) - \frac{\ln c - r\tau}{\sigma\sqrt{\tau-u}}\varphi(\frac{\ln c - r\tau - \sigma B_u}{\sigma\sqrt{\tau-u}})] d\tau\} dB_u . \tag{14}$$

On the other hand, we have

$$\int_0^T B_t I_{\{a\le S_t\le b\}}dt = \int_0^T B_t I_{\{S_t\le b\}}dt - \int_0^T B_t I_{\{S_t<a\}}dt\,. \tag{15}$$

From the relations (14) and (15) we easily obtain the representation (13).

According to the standard technique of integration, it is not difficult to see that for any positive constant $c>0$ we have

$$\int\sqrt{t}e^{-c/t}dt = \frac{2}{3}\left[\sqrt{t}e^{-c/t}(t-2c) - 2\sqrt{\pi}c^{3/2}erf\left(\frac{\sqrt{c}}{\sqrt{t}}\right)\right] + const\,, \tag{16}$$

$$\int\frac{t}{\sqrt{(t-u)^3}}e^{-c/(t-u)}dt = 2\sqrt{t-u}e^{-c/(t-u)} - \sqrt{\frac{\pi}{c}}(u-2c)erf\left(\frac{\sqrt{c}}{\sqrt{t-u}}\right) + const :=$$

$$:= h_1(t,c,u) + const \tag{17}$$

and

$$\int\frac{1}{\sqrt{t-u}}e^{-c/(t-u)}dt = \frac{2\left(\sqrt{\pi}\sqrt{c}\sqrt{u-t}\,erfi(\sqrt{c}/\sqrt{u-t})\right) + (t-u)e^{-c/(t-u)}}{\sqrt{t-u}} + const :=,$$

$$:= h_2(t,c,u) + const \tag{18}$$

where

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$$

is the error function and $erfi(z)$ is the "imaginary error function", i.e. $erfi(z) = -ierf(z)$.

Therefore, using the equality (16) with $c = \left(\frac{\ln a}{\sqrt{2}\sigma}\right)^2$ and $c = \left(\frac{\ln b}{\sqrt{2}\sigma}\right)^2$, taking into account the relations

$$erf(+\infty) = 1 \quad\text{and}\quad \lim_{t\downarrow 0}[\sqrt{t}e^{-c/t}] = 0\,, \tag{19}$$

we obtain

$$c_1(a,b,\sigma,T) := \int_0^T [\sqrt{t}\varphi(\frac{\ln c}{\sigma\sqrt{t}})]\,|_{c=a}^b\, dt = \frac{1}{\sqrt{2\pi}}\int_0^T [\sqrt{t}\exp\{-(\frac{\ln c}{\sqrt{2}\sigma})^2/t\}]\,|_{c=a}^b\, dt =$$

$$= \frac{2}{3}\left\{\left[\sqrt{T}e^{-c/T}(T-2c) - 2\sqrt{\pi}c^{3/2}erf\left(\frac{\sqrt{c}}{\sqrt{T}}\right)\right] - 2\sqrt{\pi}c^{3/2}\right\}\Bigg|_{c=(\ln a/\sqrt{2}\sigma)^2}^{(\ln b/\sqrt{2}\sigma)^2}\,. \tag{20}$$

It is clear that, in the case $\ln a \le \sigma B_u \le \ln b$, we have

$$\lim_{t\downarrow u}\Phi(\frac{\ln c - \sigma B_u}{\sigma\sqrt{t-u}}) = \begin{cases}1, & c=b;\\ 0, & c=a.\end{cases} \tag{21}$$

Therefore, using the integration by parts formula, due to the equality (17), taking into account the relations (19) and (20), it is not difficult to see that

$$\int_u^T \Phi(\frac{\ln c - \sigma B_u}{\sigma\sqrt{t-u}})\,|_{c=a}^b\, dt = \left[t\Phi(\frac{\ln c - \sigma B_u}{\sigma\sqrt{t-u}})\,|_{c=a}^b\right]\Bigg|_{t=u}^T +$$

$$+\left[\frac{\ln c-\sigma B_u}{2\sigma}\int_u^T \frac{t}{\sqrt{(t-u)^3}}\varphi(\frac{\ln c-\sigma B_u}{\sigma\sqrt{t-u}})dt\right]\Big|_{c=a}^{b}=$$

$$=T\Phi(\frac{\ln c-\sigma B_u}{\sigma\sqrt{T-u}})\,|_{c=a}^{b}-u+\left[\frac{\ln c-\sigma B_u}{2\sigma\sqrt{2\pi}}h_1(t,(\frac{\ln c-\sigma B_u}{\sigma\sqrt{2}})^2,u)\Big|_{t=u}^{T}\right]|_{c=a}^{b}=$$

$$=T\Phi(\frac{\ln c-\sigma B_u}{\sigma\sqrt{T-u}})\,|_{c=a}^{b}-u+$$

$$+\left[\frac{\ln c-\sigma B_u}{2\sigma\sqrt{2\pi}}\left(h_1(T,(\frac{\ln c-\sigma B_u}{\sigma\sqrt{2}})^2,u)-\sqrt{\frac{4\pi\sigma^2}{(\ln c-\sigma B_u)^2}}(u-(\frac{\ln c-\sigma B_u}{\sigma})^2)\right)\right]|_{c=a}^{b}:=$$

$$:=c_2(a,b,\sigma,T,u)\,. \qquad (22)$$

Analogously, using the equality (18), due to the relation (19), one can calculate

$$\int_u^T[\frac{\ln c}{\sigma\sqrt{t-u}}\varphi(\frac{\ln c-\sigma B_u}{\sigma\sqrt{t-u}})]\,|_{c=a}^{b}\,dt:=c_3(a,b,\sigma,T,u)\,. \qquad (23)$$

Combining now relations (20), (22) and (23), we obtain the following

Corollary 3. *In the case* $\mu=\sigma^2/2$ *for any real positive numbers* $a<b$ *the functional* $\int_0^T B_t I_{\{a\le S_t\le b\}}dt$ *admits the following stochastic integral representation*

$$\int_0^T B_t I_{\{a\le S_t\le b\}}dt:=-c_1(a,b,\sigma,T)+\int_0^T[c_2(a,b,\sigma,T,u)-c_3(a,b,\sigma,T,u)]dB_u\,. \qquad (24)$$

By analogy of transformations made at calculation of relations (20), (22) and (23), it is easy to calculate:

$$c_3(a,b,\sigma,T):=\int_0^T[\sqrt{t}\varphi(\frac{c}{\sqrt{t}})]\,|_{c=a}^{b}\,dt=\frac{1}{\sqrt{2\pi}}\int_0^T[\sqrt{t}\exp\{-(\frac{c}{\sqrt{2}})^2/t\}]\,|_{c=a}^{b}\,dt\,, \qquad (25)$$

$$\int_u^T\Phi(\frac{c-B_u}{\sqrt{t-u}})\,|_{c=a}^{b}\,dt=\left[t\Phi(\frac{c-B_u}{\sqrt{t-u}})\,|_{c=a}^{b}\right]\Big|_{t=u}^{T}+$$

$$+\left[\frac{c-B_u}{\sigma}\int_u^T\frac{t}{\sqrt{(t-u)^3}}\varphi(\frac{c-B_u}{\sqrt{t-u}})dt\right]\Big|_{c=a}^{b}:=c_5(a,b,T,u)\,, \qquad (26)$$

$$\int_u^T[\frac{c}{\sqrt{t-u}}\varphi(\frac{c-B_u}{\sqrt{t-u}})]\,|_{c=a}^{b}\,dt:=c_6(a,b,T,u)\,. \qquad (27)$$

Combining the relations (25), (26) and (27), we obtain the following.

Corollary 4. *For any real positive numbers* $a<b$ *the functional* $\int_0^T B_t I_{\{a\le B_t\le b\}}dt$ *have the following stochastic integral representation*

$$\int_0^T B_t I_{\{a\le B_t\le b\}}dt=-c_4(a,b,T)+\int_0^T[c_2(a,b,T,u)-c_3(a,b,T,u)]dB_u\,. \qquad (28)$$

Acknowledgement. The work is supported by the Georgia National Science Foundation.

მათემატიკა

ბროუნის ერთი არაგლუვი ფუნქციონალის სტოქასტური ინტეგრალური წარმოდგენა

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შემოთავაზებულია არაგლუვი (მალივენის აზრით) ბროუნის ფუნქციონალის სტოქასტური ინტეგრალური წარმოდგენის მიღების ერთი მეთოდი და დადგენილია ამ წარმოდგენის ინტეგრანდის ცხადი სახე. ვინაიდან კლარკ-ოკონეს კარგად ცნობილი მეთოდის გამოყენება აქ შეუძლებელია, ჩვენ ვცდილობთ არატრადიციული მეთოდის გამოყენებით მივიღოთ კლარკის ინტეგრალური წარმოდგენა ცნობილი ინტეგრანდით. ჩვენ ვიხილავთ შემთხვევას, როცა ფუნქციონალი წარმოადგენს გარკვეული სტოქასტურად არაგლუვი, კვადრატით ინტეგრებადი პროცესიდან ლებეგის ინტეგრალს (დროითი ცვლადის მიმართ), რომელიც აგრეთვე სტოქასტურად არაგლუვი ფუნქციონალია. აღმოჩნდა, რომ ფუნქციონალის სტოქასტური სიგლუვის მოთხოვნა შესაძლებელია შესუსტებულ იქნეს მხოლოდ მისი პირობითი მათემატიკური ლოდინის სიგლუვის მოთხოვნით. მიუხედავად იმისა, რომ ჩვენ მიერ განხილული ფუნქციონალის ინტეგრალქვეშა გამოსახულება აკმაყოფილებს უკანასკნელ თვისებას, მისგან მიღებულ გასაშუალოებულ (*dt*-ს მიმართ) ფუნქციონალს არ გააჩნია იგივე თვისება. პირველ ეტაპზე ჩვენ გამოგვყავს სტოქასტური ინტეგრალური წარმოდგენა ფუნქციონალის ინტეგრალქვეშა გამოსახულებისთვის, ხოლო შემდგომ, ფუბინის ტიპის სტოქასტური თეორემის საფუძველზე, ვღებულობთ სასურველ ინტეგრალურ წარმოდგენას.

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Received June, 2016