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Mathematics

Topological Invariants of Random Polynomials

Teimuraz Aliashvili*

* *Ilia State University, Tbilisi, Georgia*

(Presented by Academy Member Elizbar Nadaraya)

ABSTRACT. Random polynomials with independent identically distributed Gaussian coefficients are considered. In the case of random gradient endomorphism $F=(f,g):R^2\to R^2$ the mean topological degree is computed and the expected number of complex points is estimated. In particular, the asymptotics of these invariants are determined as the algebraic degree of F tends to infinity. We also give the asymptotic of the mean writhing number of a standard equilateral random polygon with big number of sides. *© 2016 Bull. Georg. Natl. Acad. Sci.*

Key words: random polynomial, topological degree, crossing number, writhing number

In this paper we consider pairs of random polynomials in two variables with coefficients which are normal random variables and investigate some statistical invariants of such pairs. For random polynomials of one variable, the most natural statistical invariant is the *expected number of real roots.* This invariant was investigated by M. Kac [1]. In particular, if all coefficients are independent standard Gaussian random variables M. Kac was able to find the rate of growth of the expected number of real roots as the algebraic degree of polynomial tends to infinity.

In the paper [2] the authors gave an effective formula for the average crossing number $ACN(n)$ of a standard equilateral random polygon (*SERP*) with *n* sides in three-dimensional space. This formula, in particular, gives an explicit asymptotic of this number as $n\to\infty$, which has useful application to analysis of certain qualitative phenomena in physics and biochemistry.

Notice that this result has direct consequences for random knots (knot as usual means a closed curve without self-intersections). Indeed, it is known and easy to prove that a closed polygon appearing in the model of SERP almost surely has no self-intersections. Thus the mentioned result from [2] can be considered as an estimate for the average crossing number of a random polygonal knot.

1. Much less is known about random polynomials in several variables. For example, it seems very difficult to find the expected number of real roots of $(n\times n)$ -system of random polynomial equations with independent identically distributed Gaussian coefficients. Only M. Shub and S. Smale [3] succeeded to compute this invariant for certain special distributions of coefficients. Some other developments in the spirit of [3] are summarized in [4].

These results suggested that one could try to estimate those topological invariants of random polynomials and mappings related to the real roots of polynomial systems. A natural framework for such investigations was suggested by G.Khimshiashvili [5]. As was explained in many problems of such type it is crucial to find the mean value of topological degree of a certain random endomorphism. As was conjectured in [5], this problem should be solvable for rotation invariant Gaussian distributions of coefficients introduced in [3]. This appeared possible indeed and a general result of such kind was published in [5, 6]. Similar problems were also considered in [7, 8].

All these results were concerned with the distributions introduced in [4] but there do not exist any such results in the case when all coefficients are *independent identically distributed* (i.i.d.) standard normals (N(0,1)). In this note we aim at obtaining some results for such distributions of coefficients using results of [4] and our previous results on topological invariants of planar polynomial endomorphisms [9].

By analogy with the one-dimensional case it is natural to consider a *random polynomial endomorphism* F of R^n defined by n random polynomials in n variables with fixed algebraic multi-degree $m=(m_1,\dots,m_n)$ and compute the mean *topological degree* as a function of n and m_i. For n=2 we get random endomorphisms of the plane, which besides the topological degree possess other useful numerical invariants like the number of cusps or the number of complex points. Endomorphisms of the plane are called planar endomorphisms and, following [5], we refer to them as *plends.*

The main goal of this note is to estimate the mean value of the topological degree of a random plend defined by the gradient of a random polynomial with i.i.d. central Gaussian coefficients. As was already mentioned, this means that all coefficients are real random variables and have Gaussian (normal) distribution. In the sequel the term "random polynomial" always refers to this situation. We pass now to exact formulations.

Let R_2 be the ring of real polynomials in two variables. For $P\in R_2$, let $\deg P$ denote its algebraic degree, i.e. the highest order of monomials which appear in P. Any P with $\deg P = m$ can be written as

$$P(x,y)=\sum_{k+l=0}^{m} a_{kl}x^k y^l ,$$

where appears at least one non-vanishing a_{kl} with $k+l=m$. The leader P^* is defined as the sum of monomials of highest order. Obviously it is a non-trivial binary m-form.

Suppose $a_{kl}=a_{kl}^{(\omega)}$ are real Gaussian random variables so we are given a random polynomial as above. We can also take a pair of such random polynomials (not necessarily with the same distribution of coefficients) and consider a random plend

$$F=(P,Q):R^2\to R^2$$

with these polynomials as the components. In such situation we speak of a *Gaussian random plend* and we want to estimate certain geometric characteristics of such a random plend.

As is well known, if F is proper then its (global) topological degree Deg$F$ is well-defined [5]. As one could await, a random plend almost surely (a.s.) has several nice properties of which we need here only one. It can be proved applying the same reasoning as was used in [7] to show that a random Gaussian hypersurface is almost surely smooth.

Lemma 1. *A Gaussian random plend is proper with probability one.*

For those ω for which $F(\omega)$ is not proper, we set $\mathrm{Deg}F(\omega)=0$. So we are concerned with estimating the expectation (mean value) $\mathrm{E}(\mathrm{Deg}F)$ of random variable $\mathrm{Deg}F$ and the expectation of its modulus $E(|DegP'|)$.

Theorem 1. *Let P be a Gaussian random polynomial in two variables of algebraic degree $m \geq 1$ with independent standard normal coefficients as above. Then the expectation $E(|DegP'|)$ of the absolute topological degree of its gradient P' is asymptotically equivalent to $\frac{2}{\pi}\log m$ as m tends to infinity.*

Proof. First of all, notice that it is sufficient to estimate the average topological degree of the endomorphism $(P^*)'$ defined by leaders P_x^*, P_y^* which are binary homogeneous $m-1$-forms.

Lemma 2. $E(DegP') = E\left(Deg(P^*)'\right)$.

Notice further that the zero set Z of a homogeneous polynomial P^* consists of a system of lines in R^2 passing through the origin. Their intersections with the unit circle S^1 give a finite set of points $Y = Z \cap S^1$. These points obviously appear in pairs and those pairs are in a one-to-one correspondence with the real roots of polynomial in one variable $\hat{P}$ which is obtained from P^* by dehomogenization (i.e. we divide $P^*(x,y)$ by y^m and introduce a new variable $t=\frac{x}{y}$). In other words, the number k of points in y equals $2r$, where r is the number of real roots of $\hat{P}$.

We now apply one formula which can be proved as in [9].

Lemma 3. $r = 1 - DegP'$.

Namely, first one interprets the number k as the Euler characteristic $\chi(Y)$ of the set Y. Next, according to [5], the Euler characteristic of the zero set of homogeneous polynomial P^* can be expressed through the mapping degree of its gradient by the formula

$$\chi(Y) = 2(1 - Deg(P^*)),$$

or equivalently,

$$r = 1 - Deg(P^*)' = 1 - DegP'.$$

By taking expectations of absolute values of both sides of this formula we get that the rates of growth of $E\left(|DegP'|\right)$ and $E(r)$ are equal. Thus we can estimate the expected value of absolute gradient degree by finding the expectation of the random variable equal to r. This appears possible due to the following observation which follows directly from definitions.

Lemma 4. $\hat{P}$ *is a Gaussian random polynomial of algebraic degree m with independent standard normal coefficients.*

Thus we conclude that one can compute the expected number of real roots $E(r)$ of $\hat{P}$ using Theorem 3.1 of [4]. Hence the fact that $E\left(|DegP'|\right)$ has the asymptotic indicated in the statement of the theorem follows from Theorem 2.2 of [4]. The proof is thus completed.

Actually, from the proof of Theorem 1 it follows that Lemma 3 enables us to find the exact mean value of $E(DegP')$. Indeed, to this end we can use Theorem 2.1 of [4] and to compute $E(r)$. Since coefficients of $\hat{P}$ are i.i.d. standard normals. By the formula on page 8 of [4] we get

$$E(r)=\frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{\frac{\partial^2}{\partial x\partial y}\log\frac{1-(xy)^{n+1}}{1-xy}\bigg|_{x=y=t}}\,dt\,.$$

Finally we obtain the following integral formula for the mean topological degree.

Theorem 2. $E(DegP')=1-\frac{1}{\pi}\int_{-\infty}^{+\infty}\sqrt{\frac{1}{(t^2-1)^2}-\frac{(n+1)^2t^{2n}}{(t^{2n+2}-1)^2}}dt\,.$

It should be noted that these results essentially use the specifics of gradient mappings and we are not yet able to estimate the mean topological degree for arbitrary Gaussian plend with the components of algebraic degree *m*.

2. Since the average crossing number of a knot in three-dimensional space characterizes some important topological features of its position in the space [10], this result can be considered as a contribution towards computing basic topological invariants of random polygons. As is well known, knots in three-dimensional space also possess other important topological invariants like the *writhing number* [11] and *self-linking number* [12] which are closely related to the *average crossing number.* Thus it is natural to try to compute or estimate these invariants for a standard equilateral random polygon.

Recall that a standard equilateral random polygon (*SERP*) is a widely used model for random curves and extended physical objects like polymers and *DNA* molecules [2]. In our context it can be described as follows.

Let $U=(u,v,w)$ be a three-dimensional random vector that is uniformly distributed on the unit sphere S^2, i.e., the density function of *U* is

$$\phi(U)=\begin{cases}\frac{1}{4\pi}, & \text{if}\quad |U|=\sqrt{u^2+v^2+w^2}=1.\\ 0, & \text{otherwise.}\end{cases}$$

Suppose $U_1,U_2,...,U_n$ are *n* independent random vectors uniformly distributed on S^2. An equilateral random walk of *n* steps, denoted by W_n is defined as the sequence of points in the three-dimensional space R^3:

$$X_0=0,\quad X_k=U_1+U_2+\cdots+U_k,\quad k=1,2,...,n.$$

Each X_k is called a vertex of the W_n and the line segment joining X_k and X_{k+1} is called an edge of W_n (which is of unit length). In particular, W_n becomes a polygon if $X_n=0$. In this case, it is called an equilateral random polygon and denoted by P_n. Note that the joint probability density function $f(X_1,X_2,...,X_n)$ of the vertices of P_n is simply

$$f(X_1,X_2,...,X_n)=\phi(U_1)\phi(U_2)\cdots\phi(U_n)=\phi(X_1)\phi(X_2-X_1)\cdots\phi(X_n-X_{n-1}).$$

Let X_k be the *k*-th vertex of P_n $(n\geq k>1)$. Its density function is defined by

$$f(X_k)=\iint\cdots\int\phi(X_1)\phi(X_2-X_1)\cdots\phi(X_k-X_{k-1})dX_1dX_2\cdots dX_{k-1}\,.$$

The average crossing number (*ACN*) of P_n can be defined as follows.

In [12] it is shown that the average crossing number between the non-intersecting edges l_1 and l_2 is given by

$$ACN(\gamma_1,\gamma_2)=\frac{1}{2\pi}\iint_{I\,I}\frac{\left|\dot{\gamma}_1(t),\dot{\gamma}_2(s),\gamma_1(t)-\gamma_2(s)\right|}{\left|\gamma_1(t)-\gamma_2(s)\right|^3}dtds,$$

where $\gamma_1,\gamma_2: I\to R^3$ are the arclength parametrizations of l_1 and l_2 respectively, $I=[0,1]$ and dot denotes differentiation over parameter.

For a polygonal knot *K*, one defines

$$ACN(K)=\frac{1}{2}\sum ACN(X,Y),$$

where *X, Y* are any non-consecutive sides of *K*.

Recall that the *writhing number* of a knot is defined as follows [11]. We consider its two-dimensional family of parallel projections and in each projection we count +1 or -1 for each crossing, depending on whether the overpass requires a counterclockwise or a clockwise rotation to align with the underpass. The writhing number is then the signed number of crossings averaged over all orthogonal projections on planes in R^3. It is a conformal invariant of the knot. The writhing number measures the global geometry of a closed space curve or knot.

Let γ the arclength parametrization as above and $\dot{\gamma}(t)$ denote the unit tangent vector for $t\in S^1$. The following double integral formula from [13] allows one to calculate the writhing number of two edges as above

$$W=\frac{1}{2\pi}\iint_{S^1S^1}\frac{\left\langle\dot{\gamma}_1(t),\dot{\gamma}_2(s),\gamma_1(t)-\gamma_2(s)\right\rangle}{\left|\gamma_1(t)-\gamma_2(s)\right|^3}dtds\,.$$

Correspondingly, we define

$$W(K)=\sum W(l_i,l_j),$$

where l_i and l_j are non-consecutive sides of *K* with $1\le i\le j-1\le n-1$.

Denote by $E|W(n)|$ the mean absolute value of the writhing number of a *SERP* with *n* edges. Let us say that two function $f(n)$ and $g(n)$ of *n* are asymptotically equivalent if $\lim\limits_{n\to\infty}\frac{f(n)}{g(n)}=1$.

Theorem 3. *As* $n\to\infty$, *the function* $E|W(n)|$ *is asymptotically equivalent to* $(3/16n\ln n)^{1/2}$.

The proof can be obtained by the scheme used in [5] and based on the reduction to a symmetric random walk on a real line. Indeed, according to [2] we have $E(ACN(n))=3/16n\ln n+O(n)$. Notice that from the integral formulas for the writhing number and $ACN(n)$ it follows that the only difference between these two invariants of knot is that the first one is obtained by counting each intersection in a planar projection of a knot with a sign equal to the sign of the Jacobian of the Gauss mapping. Using the calculations from [2] it is possible to show that the signs cancellation effect asymptotically leads to extracting the square root of $E(ACN(n))$, which gives the result. An intuitive explanation is that the signs behave as in one-dimensional symmetric random walk i. e., the probability of each sign on each step is $\frac{1}{2}$. Thus the mean writhing number is approximately equal to the mean absolute deviation of symmetric random walk on the real line with $M=[ACN(n)]$ steps, where [ ] denotes the integer part (entire). This explains the result, since it is well known that such mean deviation grows as $\sqrt{M}$.

მათემატიკა

შემთხვევითი პოლინომების ტოპოლოგიური ინვარიანტები

თ. ალიაშვილი*

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REFERENCES:

1. *Kac M.* (1943) Bull. Amer. Math. Soc. **49**: 314-320.
2. *Diao Y., Dobay A., Kusner R., Millet K., Stasiak A.* (2003) J. Phys. **36**: 11561-11574.
3. *Shub M., Smale S.* (1993) Progr. Math. **109**: 267-285.
4. *Edelman A., Kostlan E.* (1995) Bull. Amer. Math. Soc. **3**: 21-37.
5. *Khimshiashvili G.* (2001) Proc. A.Razmadze Math. Inst.**125**: 1-121.
6. *Aliashvili T.* (2004) Banach Center Publ. **62**: 19-28.
7. *Ibragimo I., Podkorytov S.* (1995) Dokl. Ross. Akad. Nauk. **343**: 734-736 (in Russian).
8. *Khimshiashvili G., Ushveridze A.* (1999) Bull. Georgian Acad. Sci. **159**: 385-388.
9. *Aliashvili T.* (1993) Soobshch. Akad. Nauk Gruzii **148**: 32-35 (in Russian).
10. *Dubrovin B., Novikov S., Fomenko A.* (1984) Sovremennaia Geometria, Moscow (in Russian).
11. *Agarwal P., Edelsbrunner H., Wang Y.* (2004) Discrete Comp. Geom. 32: 37-53.
12. *Freedman M., He Z.* (1991) Ann. Math. **134**: 189-229.
13. *Simon J.* (1994) Knot Theory Ramif. **3**: 299-320.

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