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Mathematics

Estimation of some Parameters of the Ornstein-Uhlenbeck Stochastic Process

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ABSTRACT. We consider Ornstein-Uhlenbeck process $x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s$, **where** $x_0 \in \mathbb{R}$, $\theta > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ **and** W_s **are Wiener process. By using the values** $(z_k)_{k \in N}$ **of the corresponding trajectories at a fixed positive moment *t*, the estimates** T_n **and** T_n^{**} **of unknown parameters** x_0 **and** θ **are constructed, where** x_0 **is an underlying asset initial price and** θ **is a rate by which these shocks dissipate and the variable reverts towards the mean in the Ornstein-Uhlenbeck's stochastic process. By using Kolmogorov's Strong Law of Large Numbers the consistence of estimates** T_n **and** T_n^{**} **are proved.** *© 2016 Bull. Georg. Natl. Acad. Sci.*

Key words: Ornstein-Uhlenbeck process, Wiener process, stochastic differential equation.

The Ornstein-Uhlenbeck process (named after Leonard Ornstein and George Eugene Uhlenbeck joint celebrated work [1]) is a Gauss-Markov stochastic process (see [2, 3]) that describes the velocity of a massive Brownian particle under the influence of friction. Over time, this process tends to drift towards its long-term mean: such a process is called mean-reverting.

In recent years, however, the Ornstein-Uhlenbeck process has appeared in finance as a model of the volatility of the underlying asset price process (see [1, 3-5]).

Note that the Ornstein-Uhlenbeck process x_t satisfies the following stochastic differential equation:

$$dx_t = \theta(\mu - x_t)dt + \sigma dW_t,$$

where $\theta > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$ are parameters and W_t denotes the Wiener process.

The solution of this stochastic differential equation has the following form

$$x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s$$

where x_0 is assumed to be constant.

The parameters in (2) have the following sense:

(i) μ represents the equilibrium or mean value supported by fundamentals (in other words, the central location);

(ii) σ is the degree of volatility around it caused by shocks;

(iii) θ is the rate by which these shocks dissipate and the variable reverts towards the mean;

(iv) x_0 is the underlying asset price at moment $t=0$ (the underlying asset initial price);

(v) x_t is the underlying asset price at moment $t>0$;

The purpose of the present article is an introduction of a new approach which allows us to construct consistent estimates for parameters x_0 and θ by the use values $(z_k)_{k\in N}$ of corresponding trajectories at a fixed positive moment *t*.

Some Auxiliary Notions and Facts from the Theory of Stochastic Differential Equations and Mathematical Statistics

Let $(\mu_\theta)_{\theta\in\vartheta}$ $(\vartheta\subseteq\mathbb{R})$ be a family of Borel probability measures defined on the real axis *R*.

Definition 1. A Borel measurable function $T_n:\mathbb{R}^n\to\vartheta$ $(\vartheta\subseteq\mathbb{R}, n\in N)$ is called a consistent estimator of a parameter $\theta\in\vartheta$ (in the sense of convergence almost everywhere) for the family $(\mu_\theta)_{\theta\in\vartheta}$ if the following condition

$$\mu_\theta^N(\{(x_k)_{k\in N}:(x_k)_{k\in N}\in\mathbb{R}^\infty \;\&\; \lim_{n\to\infty}T_n((x_k)_{1\le k\le n})=\theta\})=1$$

holds true for each $\theta\in\vartheta$, where μ_θ^N denotes the infinite power of the measure μ_θ.

Definition 2. Following [6], the family $(\mu_\theta^N)_{\theta\in\vartheta}$ is called strictly separated if there exists a family $(Z_\theta)_{\theta\in\vartheta}$ of Borel subsets of $\mathbb{R}^\infty$ such that

(i) $\mu_\theta^N(Z_\theta)=1$ for $\theta\in\vartheta$;

(ii) $Z_{\theta_1}\cap Z_{\theta_2}=\varnothing$ for all different parameters $\theta_1,\theta_2\in\vartheta$.

(iii) $\bigcup_{\theta\in\vartheta}Z_\theta=\mathbb{R}^\infty$.

In the sequel we will need the well known fact from the probability theory (see [3], p. 390).

Lemma 1 (Kolmogorov's Strong Law of Large Numbers). *Let $X_1,X_2,\cdots$ be a sequence of independent identically distributed random variables defined on the probability space $(\Omega,\Im,P)$. If these random variables have a finite expectation m, then the following condition*

$$P(\{\omega:\lim_{n\to\infty}\frac{\sum_{k=1}^n X_k(\omega)}{n}=m\})=1$$

holds true.

By the use approaches introduced in [7] one can get the validity of the following assertions.

Lemma 2. *Let's consider an Ornstein–Uhlenbeck's process x_t defined by the following stochastic differential equation:*

$$dx_t=\theta(\mu-x_t)dt+\sigma dW_t$$

where $\theta>0,\mu\in\mathbb{R}$ and $\sigma>0$ are unknown parameters and W_t denotes the Wiener process.

Then the solution of this stochastic differential equation is given by

$$x_t = x_0 e^{-\theta t} + \mu(1-e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s ,$$

where x_0 is assumed to be constant.

Lemma 3. *Under conditions of Lemma 2, the following equalities hold true:*

(i) $E(x_t) = x_0 e^{-\theta t} + \mu(1-e^{-\theta t})$;

(ii) $\mathrm{cov}(x_s, x_t) = \dfrac{\sigma^2}{2\theta}(e^{-\theta(t-s)} - e^{-\theta(t+s)})$;

(iii) $\mathrm{var}(x_s) = \dfrac{\sigma^2}{2\theta}(1-e^{-2\theta s})$.

Main Results.

Estimation of the the underlying asset initial price x_0 in an Ornstein - Uhlenbeck stochastic model can be done by the following proposition.

Theorem 1. *For* $t>0,\ x_0 \in \mathbb{R},\ \theta>0,\ \mu\in\mathbb{R}$ *and* $\sigma>0$, *let's* $\gamma_{(t,x_0,\theta,\mu,\sigma)}$ *be a Gaussian probability measure in* $\mathbb{R}$ *with the mean* $m_t = x_0 e^{-\theta t} + \mu(1-e^{-\theta t})$ *and the variance* $\sigma_t^2 = \dfrac{\sigma^2}{2\theta}(1-e^{-2\theta t})$. *Assuming that the parameters* $t>0,\ \theta>0,\ \mu\in\mathbb{R}$ *and* $\sigma>0$ *are fixed, let's denote by* γ_{x_0} *the measure* $\gamma_{(t,x_0,\theta,\mu,\sigma)}$ *for* $x_0\in\mathbb{R}$. *Let define the estimate* $T_n:\mathbb{R}^n \to \mathbb{R}$ *by the following formula*

$$T_n((z_k)_{1\le k\le n}) = e^{\theta t}\frac{\sum_{k=1}^n z_k}{n} - \mu e^{\theta t}(1-e^{-\theta t}).$$

Then we get

$$\gamma_{x_0}^N(\{(z_k)_{k\in\mathbb{N}} : (z_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty \ \&\ \lim_{n\to\infty} T_n((z_i)_{1\le i\le n}) = x_0\}) = 1$$

for $x_0\in\mathbb{R}$, *provided that* T_n *is a consistent estimator of the underlying asset price* $x_0\in\mathbb{R}$ *in the sense of convergence almost everywhere for the family of probability measures* $(\gamma_{x_0})_{x_0\in\mathbb{R}}$.

Proof. Let's consider probability space $(\Omega, \Im, P)$, where $\Omega=\mathbb{R}^\infty$, $\Im = \mathrm{B}(\mathbb{R}^\infty)$, $P=\gamma_{x_0}^N$.

For $k\in\mathbb{N}$ we consider k-th projection Pr_k defined on $\mathbb{R}^\infty$ by

$$\mathrm{Pr}_k((z_i)_{i\in\mathbb{N}}) = z_k$$

for $(z_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty$.

It is obvious that $(\mathrm{Pr}_k)_{k\in\mathbb{N}}$ is a sequence of independent Gaussian random variables with the the mean $m_t = x_0 e^{-\theta t} + \mu(1-e^{-\theta t})$ and the variance $\sigma_t^2 = \dfrac{\sigma^2}{2\theta}(1-e^{-2\theta t})$. By the use Kolmogorov's Strong Law of Large numbers (see Lemma 1), for $x_0\in\mathbb{R}$ we get

$$\gamma_{x_0}^N(\{(z_i)_{i\in\mathbb{N}} : (z_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty \ \& \ \lim_{n\to\infty} \frac{\sum_{k=1}^n \Pr_k((z_i)_{i\in\mathbb{N}})}{n} = x_0 e^{-\theta t} + \mu(1-e^{-\theta t})\}) = 1,$$

which implies

$$\gamma_{x_0}^N(\{(z_i)_{i\in\mathbb{N}} : (z_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty \ \& \ \lim_{n\to\infty} \left(e^{\theta t} \frac{\sum_{k=1}^n z_k}{n} - e^{-\theta t}\mu(1-e^{-\theta t}) \right) = x_0\})$$

$$= \gamma_{x_0}^N(\{(z_i)_{i\in\mathbb{N}} : (z_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty \ \& \ \lim_{n\to\infty} T_n((z_i)_{1\le i\le n}) = x_0\}) = 1.$$

Theorem 2. *For* $t>0,\ x_0\in\mathbb{R},\ \theta>0,\ \mu\in\mathbb{R}$ *and* $\sigma>0$, *let's* $\gamma_{(t,x_0,\theta,\mu,\sigma)}$ *be a Gaussian probability measure in* $\mathbb{R}$ *with the mean* $m_t = x_0e^{-\theta t} + \mu(1-e^{-\theta t})$ *and the variance* $\sigma_t^2 = \frac{\sigma^2}{2\theta}(1-e^{-2\theta t})$.

Assuming that the parameters $t>0,\ x_0\in\mathbb{R},\ \mu\in\mathbb{R}$ *and* $\sigma>0$ are fixed such that $x_0\neq\mu$, for $\theta>0$, let's denote by γ_θ the measure $\gamma_{(t,x_0,\theta,\mu,\sigma)}$.

Let us define the estimate $T_n^{**}:\mathbb{R}^n\to\mathbb{R}$ by the following formula

$$T_n^{**}((z_k)_{1\le k\le n}) = -\frac{1}{t}\ln\left(\frac{\frac{\sum_{k=1}^n z_k}{n} - \mu}{x_0-\mu} \right).$$

Then, for $\theta>0$ we get

$$\gamma_\theta^N(\{(z_i)_{i\in\mathbb{N}} : (z_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty \ \& \ \lim_{n\to\infty} T_n^{**}((z_i)_{1\le i\le n}) = \theta\}) = 1,$$

provided that T_n^{**} is a consistent estimator of the rate θ in the sense of convergence almost everywhere for the family of probability measures $(\gamma_\theta^N)_{\theta>0}$.

Proof. Let us consider probability space $(\Omega,\Im,P)$, where $\Omega=\mathbb{R}^\infty$, $\Im=\mathrm{B}(\mathbb{R}^\infty)$, $P=\gamma_\theta^N$.

For $k\in\mathbb{N}$ we consider k-th projection $\Pr_k$ defined on $\mathbb{R}^\infty$ by

$$\Pr_k((z_i)_{i\in\mathbb{N}}) = z_k$$

for $(z_i)_{i\in\mathbb{N}}\in\mathbb{R}^\infty$.

It is obvious that $(\Pr_k)_{k\in\mathbb{N}}$ is a sequence of independent Gaussian random variables with the mean $m_t = x_0e^{-\theta t} + \mu(1-e^{-\theta t})$ and the variance $\sigma_t^2 = \frac{\sigma^2}{2\theta}(1-e^{-2\theta t})$. By the use Kolmogorov's Strong Law of Large numbers(see Lemma 1), for $\theta>0$ we get

$$\gamma_\theta^N(\{(z_i)_{i\in\mathbb{N}} : (z_i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty \ \& \ \lim_{n\to\infty} \frac{\sum_{k=1}^n \Pr_k((z_i)_{i\in\mathbb{N}})}{n} = x_0 e^{-\theta t} + \mu(1-e^{-\theta t})\}) = 1,$$

which implies

$$\gamma_\theta^N(\{(z_i)_{i\in\mathbb{N}}:(z_i)_{i\in\mathbb{N}}\in\mathbb{R}^\infty \;\&\; \lim_{n\to\infty}-\frac{1}{t}\ln\left(\frac{\frac{\sum_{k=1}^n z_k}{n}-\mu}{x_0-\mu}\right)=\theta\})$$

$$=\gamma_\theta^N(\{(z_i)_{i\in\mathbb{N}}:(z_i)_{i\in\mathbb{N}}\in\mathbb{R}^\infty \;\&\; \lim_{n\to\infty}T_n^{**}((z_i)_{1\le i\le n})=\theta\})=1.$$

Simulation of the Ornstein - Uhlenbeck Stochastic Process

The simulation of the Ornstein-Uhlenbeck process can be obtained as follows:

$$x_t = x_0 e^{-\theta t}+\mu(1-e^{-\theta t})+\frac{\sigma}{\sqrt{2\theta}}e^{-\theta t}W_{e^{2\theta t}-1},$$

where W_t denotes Wiener process.

Wiener (1923) gave a representation of a Brownian path in terms of a random Fourier series. If $(\xi_n)_{n\in\mathbb{N}}$ is the sequence of independent standard Gaussian random variables, then

$$W_t=\xi_0 t+\sqrt{2}\sum_{n=1}^{\infty}\xi_n\frac{\sin\pi nt}{\pi n}$$

represents a Brownian motion on [0,1].

Following Karhunen -Loeve well known theorem (see [8]), the scaled process

$$\sqrt{c}W_t\left(\frac{t}{c}\right)$$

is a Brownian motion on $[0,c]$.

In our simulation we use MatLab command **random('Normal', 0, 1, p, q)** which generates normally distributed sequences $(\xi_n^{(k)})_{1\le n\le q}(1\le k\le p)$.

Note that

$$W^{(k)}_{e^{2\theta t}-1}=\xi_0^{(k)}(e^{2\theta t}-1)+\sqrt{2}\sum_{n=1}^{\infty}\xi_n^{(k)}\frac{\sin\pi n(e^{2\theta t}-1)}{\pi n}$$

will be the value of the Wiener's k-th trajectory at moment $e^{2\theta t}-1$ for $k\in\mathbb{N}$.

Hence the value z_k of the Ornstein-Uhlenbeck's k-th trajectory at moment t will be

$$z_k=x_0e^{-\theta t}+\mu(1-e^{-\theta t})+\frac{\sigma}{\sqrt{2\theta}}e^{-\theta t}\left(\xi_0^{(k)}(e^{2\theta t}-1)+\sqrt{2}\sum_{n=1}^{\infty}\xi_n^{(k)}\frac{\sin\pi n(e^{2\theta t}-1)}{\pi n}\right)$$

for $k\in\mathbb{N}$.

In our simulation we consider

$$z_k=x_0e^{-\theta t}+\mu(1-e^{-\theta t})+\frac{\sigma}{\sqrt{2\theta}}e^{-\theta t}\left(\xi_0^{(k)}(e^{2\theta t}-1)+\sqrt{2}\sum_{n=1}^{800}\xi_n^{(k)}\frac{\sin\pi n(e^{2\theta t}-1)}{\pi n}\right)$$

for $1\le k\le 100$.

Below we present some numerical results obtaining by using MatLab and Microsoft Excel. In our simulation:

(i) n denotes the number of trials;

(ii) $x_0 = 3$ is the underlying asset initial price;

(iii) $\mu = -3$ is the equilibrium or mean value supported by fundamentals;

(iv) $\sigma = 1$ is the degree of volatility around it caused by shocks;

(v) $\theta = 0.5$ is the rate by which these shocks dissipate and the variable reverts towards the mean;

(vi) $t = 0.5$ is the moment of the observation on the Ornstein-Uhlenbeck's trajectories;

(vii) z_k is the value of the Ornstein-Uhlenbeck's k-th trajectory at moment $t = 0.5$ (see, Fig. and Table 1).

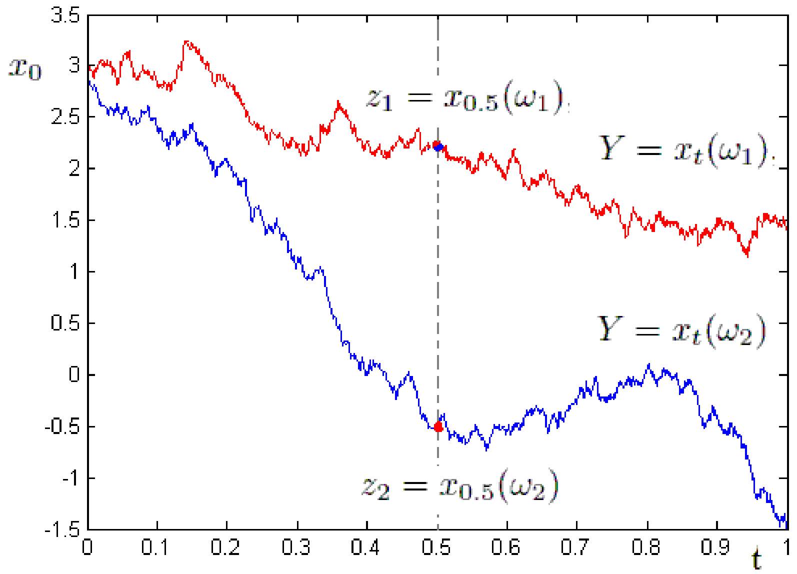

Fig. Ornstein - Uhlenbeck's two trajectories when $x_0 = 3,\ \mu = -3,\ \sigma = 1,\ \theta = 0.5$.

Table 1. The value z_k of the Ornstein-Uhlenbeck's k-th trajectory at moment t = 0.5 when x_0=3, μ = –3, σ =1, θ =0.5

k	z_k	k	z_k	k	z_k	k	z_k	k	z_k
1	2.7082	21	1.2571	41	1.2185	61	1.9426	81	2.7082
2	2.0594	22	2.1261	42	3.0131	62	1.453	82	2.0594
3	2.1303	23	2.6017	43	2.0324	63	1.6666	83	2.1303
4	2.3939	24	0.7975	44	1.8216	64	1.2806	84	2.3939
5	2.641	25	1.9225	45	1.1374	65	1.3268	85	2.641
6	1.1519	26	1.8187	46	2.7327	66	1.4312	86	1.1519
7	1.6549	27	1.8187	47	2.3649	67	2.7034	87	1.6549
8	1.2017	28	1.1202	48	1.3785	68	1.227	88	0.6265
9	1.261	29	0.3467	49	2.6211	69	1.0065	89	1.2017
10	0.8576	30	1.2734	50	1.258	70	0.7277	90	1.261
11	1.3968	31	2.6075	51	1.5606	71	1.3327	91	0.8576
12	2.8304	32	1.1872	52	2.0278	72	1.3528	92	1.3968
13	1.1969	33	1.9519	53	1.3095	73	2.102	93	2.8304
14	3.0469	34	1.9615	54	1.8024	74	1.1705	94	1.1969
15	0.7784	35	1.6775	55	1.62	75	1.162	95	3.0469
16	1.6111	36	2.5195	56	0.9569	76	0.9056	96	0.7784
17	1.1053	37	1.894	57	0.8123	77	0.6306	97	1.6111
18	1.2695	38	0.9174	58	0.9781	78	0.3304	98	1.1053
19	1.2756	39	1.5291	59	1.9541	79	1.0314	99	1.2695
20	1.711	40	1.3806	60	1.4921	80	1.9173	100	1.2756

Table 2. The value of the statistic T_n for the sample $(z_k)_{1\le k\le n}$ $(n = 5i : 1 \le i \le 20)$ from Table 1

n	T_n	x_0	n	T_n	x_0
5	3.916479949	3	55	3.050042943	3
10	3.171013312	3	60	2.999422576	3
15	3.189798604	3	65	2.985749187	3
20	3.053011377	3	70	2.963503799	3
25	3.059916865	3	75	2.944638775	3
30	2.933186125	3	80	2.891140712	3
35	2.98020897	3	85	2.951454785	3
40	2.978720876	3	90	2.918940576	3
45	3.005595171	3	95	2.936244133	3
50	3.056170079	3	100	2.90958959	3

Table 3. The value of the statistic T_n^{**} for the sample $(z_k)_{1\le k\le n}$ $(n = 5i : 1 \le i \le 20)$ from Table 1

n	T_n^{**}	θ	n	T_n^{**}	θ
5	0.215705016	0.5	55	0.483388203	0.5
10	0.44379283	0.5	60	0.500192489	0.5
15	0.437713842	0.5	65	0.504755927	0.5
20	0.482407151	0.5	70	0.512202556	0.5
25	0.480126781	0.5	75	0.518539409	0.5
30	0.522396228	0.5	80	0.536619648	0.5
35	0.50660792	0.5	85	0.516247561	0.5
40	0.507105654	0.5	90	0.527203992	0.5
45	0.498135817	0.5	95	0.521365679	0.5
50	0.481363743	0.5	100	0.530366173	0.5

Remark 1. By using the results of calculations placed in Tables 2-3, we see that both consistent estimators T_n and T_n^{**} work successfully.

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მათემატიკა

ორშტეინ-ულენბეკის შემთხვევითი პროცესის ზოგიერთი პარამეტრის შეფასება

ლ. ლაბაძე

საქართველოს ტექნიკური უნივერსიტეტი, მათემატიკის დეპარტამენტი, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

განხილულია ფინანსებში ძირითადი აქტივის ფასის ცვლილების აღმწერი ორშტეინ-ულენბეკის შემთხვევითი პროცესი $x_t = x_0 e^{-\theta t} + \mu(1-e^{-\theta t}) + \sigma\int_0^t e^{-\theta(t-s)} dW_s$, სადაც $x_0 \in \mathbb{R}$, $\theta > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ და W_s ვინერის პროცესია. ტრაექტორიების დროის ფიქსირებულ დადებით მომენტში $(z_k)_{k\in N}$ დაკვირვებების საშუალებით აგებულია x_0 და θ პარამეტრების შეფასებები T_n და T_n^{**}, სადაც x_0 წარმოადგენს ძირითადი აქტივის საწყის ფასს და θ წარმოადგენს იმ სიჩქარეს, რომლითაც ძირითადი აქტივის ფასების ცვლილებაზე მიყენებული შოკები მიმოიფანტება და უბრუნდება საშუალოს. კოლმოგოროვის დიდ რიცხვთა გაძლიერებული კანონის გამოყენებით დამტკიცებულია T_n და T_n^{**} შეფასებების ძალდებულობა.

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