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Mathematics

On One Discrete Model of the Financial Market

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ABSTRACT. In the paper it is considered one particular model of the discrete financial market with two bonds and one stock. The interest rate dependent on time and related martingale measure are constructed. The relationship between martingale measure and arbitrage opportunity of financial market is established. An illustrative two-step numerical example of calculation of the European call option is given. *© 2016 Bull. Georg. Natl. Acad. Sci.*

Key words: financial market, martingale measure, arbitrage, standard call option

1. Let us consider financial market in discrete time $(B,S)=(B_n,S_n)$, $n=0,1,\dots,N$, in which prices of the assets B and S are given by the following recurrent equalities

$$B_n=(1+r_n)B_{n-1},\quad B_0>0, \tag{1}$$

$$S_n=(1+\rho_n)S_{n-1},\quad S_0>0, \tag{2}$$

where B_n bond price (bank account) satisfies the following relation

$$B_n=B_n^{(1)}+B_n^{(2)},$$

bonds $B_n^{(1)}$ and $B_n^{(2)}$ are defined by equalities

$$B_n^{(1)}=\left(1+r^{(1)}\right)B_{n-1}^{(1)}, \tag{3}$$

$$B_n^{(2)}=\left(1+r^{(2)}\right)B_{n-1}^{(2)}, \tag{4}$$

In formulas (3) and (4) interest rates $r^{(1)}>0$ and $r^{(2)}>0$ are the constants. In (2), which defines price of the stock S_n, ρ_n is the sequence of independent identically distributed random variables, that take only two values a and b, $a<b$, with probabilities $p>0$ and $1-p$ respectively, [1, 2]. At that $a<r^{(1)}<b$, $a<r^{(2)}<b$, $B_0^{(1)}\neq B_0^{(2)}$. As for the interest rate r_n, it is defined for each n with the following representation

$$r_n = \frac{r^{(1)}B_{n-1}^{(1)} + r^{(2)}B_{n-1}^{(2)}}{B_{n-1}^{(1)} + B_{n-1}^{(2)}}. \tag{5}$$

In the model (1), (2) of a discrete time financial market, B is the riskless asset, while S – risky asset. It is assumed that $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$ is a stochastic basis, where $\mathcal{F}_n = \sigma\{S_0, \ldots, S_n\}$ – is the minimal σ -algebra generated by $S_0, \ldots, S_n$.

2. Consider two dimensional stochastic sequence $\pi = \pi_n = (\beta_n, \gamma_n)$, where γ_n is $\mathcal{F}_{n-1}$-measurable. The elements β_n and γ_n are quantities of the assets *B* and *S*, respectively, at the moment *n*. The pair (β_n, γ_n) is called investment strategy or portfolio. Investment capital of the portfolio $\pi_n = (\beta_n, \gamma_n)$ is the stochastic sequence $X^\pi = (X_n^\pi)$, $n = 0, 1, \ldots, N$, which is given with the following relation

$$X_n^\pi = \beta_n B_n + \gamma_n S_n.$$

The class of strategies $\pi_n = (\beta_n, \gamma_n)$, which satisfies the condition

$$\Delta\beta_n B_{n-1} + \Delta\gamma_n S_{n-1} = 0,$$

where $\Delta\beta_n = \beta_n - \beta_{n-1}$, $\Delta\gamma_n = \gamma_n - \gamma_{n-1}$, is called self-financing and is denoted by *SF*. The capital of self-financing portfolio $\pi_n = (\beta_n, \gamma_n)$ admits the representation

$$X_n^\pi = X_0^\pi + \sum_{k=1}^{n} (\beta_k \Delta B_k + \gamma_k \Delta S_k),$$

where $\Delta B_k = B_k - B_{k-1}$, $\Delta S_k = S_k - S_{k-1}$.

Self-financing portfolio $\pi \in SF$ is referred as a strategy with arbitrage if it implements the arbitrage opportunity of the market in the following sense:

(a) $X_0^\pi = x \le 0$,

(b) $X_N^\pi(\omega) \ge 0$ for all $\omega \in \Omega$,

(c) $X_N^\pi(\omega) > 0$ for some $\omega \in \Omega$.

We denote the class of such portfolios by SF_{arb}. Arbitrage or arbitrage-free property of financial (B, S) market depends on whether the class SF_{arb} is empty or not.

Let us consider the probability measure P^* equivalent to *P*. The measure P^* is called as martingale measure or risk-natural measure, if the stochastic sequence S_n / B_n, $n \le N$, is a martingale with respect to P^*. We denote the class of such measures by $\mathbb{P}^*$. It is interesting to find on the (B, S) market (1), (2) the martingale condition.

3. The following theorem defines a martingale criterion for the measure $P^* \in \mathbb{P}^*$.

Theorem 1. *Let in the model* (1), (2) *of financial* (B, S) *-market, the deterministic sequence* $r = (r_n)$, $n = 0, 1, \ldots, N$, *satisfies condition* $r_n > -1$. *Then with respect to probability measure* P^* *we have*

$$R_n = \frac{S_n}{B_n} \text{ is a martingale} \Leftrightarrow \sum_{k=0}^{n} (\rho_k - r_k) \text{ - is a martingale,}$$

where r_k *is defined by relation* (5).

Proof. We introduce the following notations [3]

$$U_n = \sum_{k=0}^{n} r_k, \quad V_n = \sum_{k=0}^{n} \rho_k.$$

With these values prices of bonds B_n and stocks S_n can be written in the form of stochastic exponents

$$B_n = B_0 \mathcal{E}_n(U), \quad S_n = S_0 \mathcal{E}_n(V),$$

where stochastic exponents

$$\mathcal{E}_n(U) = \prod_{k=1}^{n} (1 + \Delta U_k), \quad \mathcal{E}_0(U) = 1,$$

$$\mathcal{E}_n(V) = \prod_{k=1}^{n} (1 + \Delta V_k), \quad \mathcal{E}_0(V) = 1.$$

Further, according to the stochastic exponents properties and by Theorem 2.5 of [3], we can write

$$R_n = \frac{S_n}{B_n} = R_0 \mathcal{E}_n(V) \mathcal{E}_n^{-1}(U) = R_0 \mathcal{E}_n\left(\sum_{k=1}^{n} \frac{\Delta V_k - \Delta U_k}{1 + \Delta U_k} \right)$$

Hence, by Theorem 2.5 of [3] it follows that R_n is a local martingale if and only if when the sequence

$$Q_n = \sum_{k=0}^{n} (\rho_k - r_k)$$

is a local martingale. Theorem 1 is proved.

Of course, it is of interest to find out the relationship between arbitration of (B,S)-market and the martingale property of probability measure $P^* \in \mathbb{P}^*$. The following theorem, called the fundamental theorem of financial mathematics, gives an answer to the question.

Theorem 2. *Suppose that in the model* (1), (2) *of financial* (B,S) *-market, deterministic sequence* $r = (r_n)$ *is such, that* $r_n > -1$, $n \in \mathbb{N}$. *Then*

$$\mathbb{P}^* \neq \varnothing \Leftrightarrow SF_{arb} = \varnothing.$$

The proof of the implication $(\Rightarrow)$**.** Let $P^* \in \mathbb{P}^*$. Then for any self-financing strategy $\pi \in SF$ we have

$$\Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n \Delta S_n = r_n X_{n-1}^\pi + \gamma_n S_{n-1} (\rho_n - r_n).$$

Hence, U_n is deterministic, it follows from martingale property of P^* and Theorem 1, that if $X_0^\pi = 0$, then

$$E^* X_n^\pi = \mathcal{E}_n(U) E^* X_0^\pi = 0, \quad n \in \mathbb{N}. \tag{6}$$

Suppose opposite that $SF_{abr} \neq \varnothing$ and $\pi \in SF_{abr}$. Then, since the measures P and P^* are equivalent, we obtain $E^* X_n^\pi > 0$, which contradicts with (6). Implication $(\Rightarrow)$ is proved.

The proof of implication $(\Leftarrow)$**.** Let $SF_{abr} = \varnothing$. Note that the proof of this fact, as in [3], is reduced to the proof of the following equation

$$E^*\left(\frac{S_\tau}{B_\tau}-\frac{S_0}{B_0}\right)=0, \tag{7}$$

where $\tau=\tau(\omega)$ is the stopping time with values $0,1,\dots,N$, and $\left(S_n/B_n, \mathcal{F}_n, P^*\right)$ is a martingale. Indeed, we can chose the stopping time τ^* and construct the sequence $\pi^*=\left(\pi_n^*\right)$, that $E^* X_N^{\pi^*}=0$ [3]. Then it is easy to see, that

$$0=E^* X_N^{\pi^*}=E^*\left(\beta_N^* B_N+\gamma_N^* S_N\right)=B_N E^*\left(\frac{S_{\tau^*}}{B_{\tau^*}}-\frac{S_0}{B_0}\right).$$

Noting that $B_N \neq 0$, it proves (7), implication $(\Leftarrow)$ and consequently Theorem 2.

Theorem 3. *Let in the model* (1), (2) *of financial market* (B,S), $r_n>-1$, $n\in\mathbb{N}$. *Then the measure*

$$P^*=\frac{r_n-a}{b-a}$$

is a martingale measure, *where* r_n *is defined by* (5).

Proof. We have

$$E\left(\rho_n-r_n\right)=a(1-p)+bp-r_n=(b-a)p-\left(r_n-a\right).$$

From this equality we determine the measure p^* by

$$p^*=\frac{r_n-a}{b-a}$$

and let P^* be the probability measure corresponding to value p^*. Then easy to see, that the sequence $\left(m_n, \mathcal{F}_n, P^*\right)$, $n\in\mathbb{N}$, where

$$m_n=\sum_{k=1}^{n}\left(\rho_k-r_k\right)$$

is a martingale. Theorem 3 is proved.

4. Here is a numerical example.

Consider the model (1), (2) and suppose, that there are the following initial data:

$$\text{(I)}\quad \begin{aligned} &B_0^{(1)}=30,\quad r^{(1)}=\frac{1}{5},\quad B_0^{(2)}=20,\quad r^{(2)}=\frac{1}{2},\\ &S_0=100,\quad a=-\frac{2}{5},\quad b=\frac{3}{5},\quad k=100.\end{aligned}$$

Example. Let (1), (2) be the model of financial (B,S) market and numerical data (I) are given. We solve the two-step problem of calculation of the European standard call option

Solution. We have $N=2$, $n=0,1,2$, and pay-off is of the following form

$$f_2=f\left(S_2\right)=\max\left(S_2-K,0\right).$$

Let us calculate the parameters of two-step tree. Then

$$1+r^{(1)}=\frac{6}{5},\quad r_1=\frac{8}{25},\quad p_1^*=\frac{18}{25},\quad 1+r_1=\frac{33}{25};$$

$$B_1^{(1)}+B_1^{(2)}=36+30=66=\frac{33}{25}\cdot 50=66\,.$$

$$r_2=\frac{37}{110},\quad 1+r_2=\frac{147}{110},\quad p_2^*=\frac{81}{110};$$

$$B_2^{(1)}+B_2^{(2)}=\frac{216}{5}+45=\frac{441}{5}=\frac{147}{110}\cdot 66=\frac{441}{5}\,.$$

$$C_{10}=\left(1+r_2\right)^{-1}\left[p_2^*f_{21}+\left(1-p_2^*\right)f_{20}\right]=0,$$
$$C_{11}=\left(1+r_2\right)^{-1}\left[p_2^*f_{22}+\left(1-p_2^*\right)f_{21}\right]=\frac{27\cdot 156}{49},$$
$$C_2=\left(1+r_1\right)^{-1}\left[p_1^*C_{11}+\left(1-p_1^*\right)C_{10}\right]=\frac{6\cdot 27\cdot 156}{11\cdot 49}.$$

In the moment n=0 we construct the minimal hedge $\pi_1^*=\left(\beta_1^*,\gamma_1^*\right)$:

$$\beta_1^*=\frac{(1+b)C_{10}-(1+a)C_{11}}{(1+r_1)(b-a)\left(B_0^{(1)}+B_0^{(2)}\right)}=-\frac{3\cdot 81\cdot 156\cdot 25}{33\cdot 50\cdot 5\cdot 147},$$
$$\gamma_1^*=\frac{C_{11}-C_{10}}{(b-a)S_0}=\frac{81\cdot 156}{100\cdot 147}.$$

The corresponding to this hedge initial capital equals

$$X_0^{\pi^*}=\beta_1^*\left(B_0^{(1)}+B_0^{(2)}\right)+\gamma_1^*S_0=\frac{18\cdot 81\cdot 156}{33\cdot 147}=C_2\,.$$

For the illustration consider only possible trajectory $S_0\to S_{21}=160$.

In the moment n=1 we construct the minimal hedge $\pi_2^*=\left(\beta_2^*,\gamma_2^*\right)$. For the considered trajectory we have:

$$\beta_2^*=\frac{(1+b)f_{2,1}-(1+a)f_{2,2}}{(1+r_2)(b-a)\left(B_1^{(1)}+B_1^{(2)}\right)}=-\frac{3\cdot 156\cdot 110}{5\cdot 147\cdot 66},$$
$$\gamma_2^*=\frac{f_{2,2}-f_{2,1}}{(b-a)S_{1,1}}=\frac{156}{160}.$$

The corresponding to this hedge capital of investor in the moments n=1 and n=2 is: in the moment n=1 for C_{11} we have

$$X_1^{\pi^*}=\beta_2^*\left(B_1^{(1)}+B_1^{(2)}\right)+\gamma_2^*S_{1,1}=C_{1,1}=\frac{27\cdot 156}{49},$$

in the moment n=2 for $S_{2,2}$ and $S_{2,1}$

$$X_2^{\pi^*} = \beta_2^*\left(B_2^{(1)} + B_2^{(2)}\right) + \gamma_2^* S_{2,2} = f_{2,2} = 156,$$

$$X_2^{\pi^*} = \beta_2^*\left(B_2^{(1)} + B_2^{(2)}\right) + \gamma_2^* S_{2,1} = f_{2,1} = 0.$$

Thus, for the considered trajectory two-step problem of calculation of the European standard call option is solved.

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ფინანსური ბაზრის ერთი დისკრეტული მოდელის შესახებ

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