საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 11, №1, 2017
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 11, no. 1, 2017

Mathematics

On the Complex Points of Random Polynomials

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ABSTRACT. Random polynomials with independent identically distributed Gaussian coefficients are considered. In the case of random gradient endomorphism $F = (f, g): R^2 \to R^2$ the expected number of complex points is estimated. In particular, the asymptotic of this invariants determined as the algebraic degree of F tends to infinity. We also obtained a lower estimate for the mean Coulomb energy of a standard equilateral random polygon. *© 2017 Bull. Georg. Natl. Acad. Sci.*

Key words: Random polynomial, Gaussian distribution, complex point, Coulomb energy

In this paper we will discuss pairs of random polynomials in two variables with coefficients which are normal random variables and investigate some statistical invariants of such pairs.Similar problems were considered in previous paper. All these results were concerned with the distributions introduced in [1] but there do not exist any such results in the case when all coefficients are *independent identically distributed* (i.i.d.) standard normals (N(0,1)). In this note we aim at obtaining some results for such distributions of coefficients using results of [1] and our previous results.

Remind some notations from probability theory.

Recall that if ¶ is random variable with Gaussian (normal) density

$$f_{\P}(x) = \frac{1}{\sqrt{2®}\,\pm} e^{-\frac{(x-a)^2}{2\pm^2}},$$

$\pm > 0$, $-\infty < a < +\infty$, then the parameters a and $\pm$ are

$$a = E\P, \quad \pm^2 = D\P$$

expectation and variance, correspondingly.

If $(\P, ¦)$ is a pair of random variables, then the value

$$\mathrm{cov}(\P, ¦) = E[(\P - E\P)(¦ - E¦)]$$

is called covariation of ¶ and ¦. If $\mathrm{cov}(\P, ¦) = 0$, then ¶ and ¦ are called non-correlated. Variance $D\P$ is defined as $\mathrm{cov}(\P, \P) = D\P$. Coefficients of correlation are defined as

$$°(¶,¦) = \frac{\text{cov}(¶,¦)}{\sqrt{D¶ \cdot D¦}} \quad .$$

Since we are going to deal with random endomorphisms of the plane, we write down that explicitly the *2-dimensional normal density* is

$$f_{¶¦}(x,y) = \frac{1}{2®±_1±_2\sqrt{1-°}} \exp\left\{-\frac{1}{2(1-°)^2}\left[\frac{(x-a_1)^2}{±_1^2} - 2° \cdot \frac{(x-a_1)\cdot(y-a_2)}{±_1±_2} + \frac{(y-a_2)^2}{±_2^2}\right]\right\}.$$

It is characterized by five parameters $a_1, a_2, ±_1, ±_2, °$, where $|a_1| < \infty$, $|a_2| < \infty, ±_1 > 0 >0 \ ±_2 > 0$, $|°| < 1$. They are:

$$a_1 = E¶, \quad a_2 = E¦,$$

$$±_1^2 = D¶, \quad ±_2^2 = D¦,$$

$$° = °(¶,¦).$$

The normal distribution is

$$P(¶ \in B) = \Phi_{a,±^2}(B) = \frac{1}{\sqrt{2®}±} \cdot \int_B e^{-\frac{(u-a)^2}{2±^2}} du\ ,$$

if $a = 0$ and $± = 1$ then we have standard distribution $\Phi_{0,1}$ (denoted by $\Phi(x)$)

$$\Phi(x) = \Phi_{0,1}(-\infty, x) = \frac{1}{\sqrt{2®}} \cdot \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du\ .$$

1. We add some remarks on another invariant of random polynomial endomorphisms mentioned in our previous paper, namely, the expectation $E(c(F))$, where $c(F)$ is the number of *complex points* of F. A natural setting in this context is to consider random polynomial endomorphisms with fixed algebraic degrees of the components.

Definition. *A point $p \in R^2$ is called a complex point of F if the tangent plane $T_{(p,F(p))}\Gamma_F$ to the graph Γ_F of F in C^2 is a complex line.*

In other words, we estimate $E(C(\Gamma_F))$, where $C(\Gamma_F)$ is the number of complex points on the graph $\Gamma_F \subset C^2 = R^2 \times R^2$. For our purposes it is useful to give an analytic description of complex points.

Lemma. *Complex points are exactly the zeros of the polynomial endomorphism $\frac{\partial F}{\partial \bar{z}}$, where*

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

So, it becomes possible to apply results from [2], which enables one to compute the number of complex points in concrete cases and obtain some general estimates in terms of the algebraic degree of F. Moreover, the algebraic number of complex points can be computed as the local topological degree at infinity of $\partial F / \partial \bar{z}$, so we can estimate its mean value using the results presented in previous paper.

So, if $F = (f,g) : R^2 \to R^2$ is a polynomial endomorphism, then $\bar{\partial} F = \partial_{\bar{z}} F : R^2 \to R^2$ and the set of complex points of F coincides with $\bar{\partial} F^{-1}(0)$. We are interested in the case when the coefficients of polynomial endomorphisms are i.i.d. standard normals. More precisely, consider $F^{(µ)} = (f^{(µ)}, g^{(µ)}) : R^2 \to R^2$, where

$$f^{(\mu)} = \sum_{0 \le k+1 \le n} a_{kl}^{(\mu)} x^k y^l \text{ and } g^{(\mu)} = \sum_{0 \le k+1 \le n} b_{kl}^{(\mu)} x^k y^l$$

are the random polynomials. Let the coefficients $a_{kl}^{(\mu)}$ and $b_{kl}^{(\mu)}$ $(0 \le k+1 \le n)$ be independent standard normals. In order to estimate the number of complex points $\mathbf{C}(f, g)$ one can express $E\left((\bar{\partial}F)^{-1}(0)\right)$ by covariance matrix and moment curve as in [3]. To this end, we observe that the functions appearing in Cauchy-Riemann conditions have the form:

$$f_x = \sum_{0 \le k+1 \le n} k \cdot a_{kl} x^{k-1} y^l, \quad f_y = \sum_{0 \le k+1 \le n} l \cdot a_{kl} x^k y^{l-1},$$

$$g_x = \sum_{0 \le k+1 \le n} k \cdot b_{kl} x^{k-1} y^l, \quad g_y = \sum_{0 \le k+1 \le n} l \cdot b_{kl} x^k y^{l-1}.$$

Here a_{kl} and b_{kl} are the same as $a_{kl}^{(\mu)}$ and $b_{kl}^{(\mu)}$ above. So, the polynomials

$$f_x - g_y = \sum_{0 \le k+1 \le n} \left(k \cdot a_{kl} x^{k-1} y^l - l \cdot b_{kl} x^k y^{l-1}\right), \; f_y + g_x = \sum_{0 \le k+1 \le n} \left(l \cdot a_{kl} x^k y^{l-1} + k \cdot b_{kl} x^{k-1} y^l\right)$$

have coefficients, which are central Gausssian random variables with variances which can be computed using the fact that $D(ka + lb) = k^2 D(a) + l^2 D(b)$. Moreover, one can also compute the pairwise covariations of the coefficients of the above two polynomials. Therefore, we obtain a new polynomial endomorphism

$$\left(f_x - g_y, f_y + g_x\right) : R^2 \to R^2$$

with a multivariate normal distribution of coefficients with covariation matrix C. Now using the Theorem 7.1 of [3] we can find the expected number of complex points by simply substituting the matrix C in the integral formula on page 29 of [2]. Now using the estimate given on page 30 of [1] we can conclude that E(c(F)) grows no faster than $Const(\log m)^2$ as m tends to infinity.

2. Recall that the *Coulomb energy* of a polygonal knot is defined as follows [4]. For disjoint line segments X, Y in R^3 the energy is equal to

$$I(X,Y) = \iint_{X\,Y} \frac{dydx}{\|x-y\|^2}.$$

Then, for a polygon *K*, one defines:

$$I(K) = \sum I(X,Y)$$

(the sum is over all non-consecutive segments X, Y of *K*).

In order to relate the energy with the average crossing number recall that the energy of a pair of smooth paths $\yen_1, \yen_2 : I \to R^3$ can be computed as:

$$I(\yen_1, \yen_2) = \iint_{I\,I} \frac{\left|\dot{\yen}_1(t)\right| \cdot \left|\dot{\yen}_2(s)\right|}{\left|\yen_1(t) - \yen_2(s)\right|^2} dudv.$$

Using this and the evident inequalities

$$\iint_{I\,I} \frac{\left\langle \dot{\yen}_1(t) \times \dot{\yen}_2(s), \yen_1(t) - \yen_2(s) \right\rangle}{\left|\yen_1(t) - \yen_2(s)\right|^3} dtds \le \iint_{I\,I} \frac{\left|\dot{\yen}_1(t) \times \dot{\yen}_2(s)\right|}{\left|\yen_1(t) - \yen_2(s)\right|^2} dtds \le \iint_{I\,I} \frac{\left|\dot{\yen}_1(t)\right| \left|\dot{\yen}_2(s)\right|}{\left|\yen_1(t) - \yen_2(s)\right|^2} dtds,$$

we get

$$I(¥_1, ¥_2) \geq 4® \cdot ACN(¥_1, ¥_2).$$

Now for a polygonal knot K it is easy to show that $I(K) \geq 4® \cdot ACN(K)$.

Finally, let $E(n)$ denote the mean value of Coulomb energy of a SERP P_n with n sides.

Theorem. *For sufficiently big n, one has* $\lim_{n\to\infty} \frac{4E(n)}{3® \, n \ln n} \geq 1.$

From the above inequality it follows that $E(n) \geq 4® \, E(ACN(n))$. Thus, our result follows from the main result of [5].

მათემატიკა

შემთხვევითი პოლინომების კომპლექსური წერტილების შესახებ

თ. ალიაშვილი

ილიას სახელმწიფო უნივერსიტეტი, საბუნებისმეტყველო მეცნიერებათა და საინჟინრო ფაკულტეტი, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

განხილულია შემთხვევითი პოლინომები, რომელთა კოეფიციენტები განაწილებულია გაუსის კანონით. გრადიენტული ენდომორფიზმის შემთხვევაში გამოთვლილია კომპლექსური წერტილების საშუალოს შეფასება. კერძოდ, მიღებულია ამ ინვარიანტის ასიმპტოტიკა, როცა ალგებრული ხარისხის მაჩვენებელი მიისწრაფვის უსასრულობისკენ. მიღებულია აგრეთვე, სტანდარტული შემთხვევითი ტოლგვერდა პოლიგონებისთვის კულონის ენერგიის საშუალოს შეფასება ქვემოდან.

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Received October, 2016