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Mathematics

On a Linear Conjugation Boundary Value Problem for Piecewise-Continuous Coefficients

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ABSTRACT. We consider a boundary value problem of linear conjugation with the boundary condition

$$¤^{+}(t)=G(t)¤^{-}(t)+g(t), \qquad t\in\Gamma$$

where Γ is a simple closed Carleson line, $G(t)$ and $g(t)$ are given functions on Γ, $G(t)$ is piecewise continuous, $0<m<|G(t)|<M<\infty$, and $g(t)\in L^{p(\cdot)}(\Gamma)$. As usual $L^{p(\cdot)}(\Gamma)$ denotes the Lebesgue space with variable exponent. The sought function is representable by the Cauchy integral with the principal part at infinity and a density from $L^{p(\cdot)}(\Gamma)$. Additional restrictions are imposed at the discontinuity points of the function $G(t)$ as in the works of other authors, while in the present paper they are lesser than those of other authors. The solutions of the problem are written explicitly.

Key words: boundary value problem, weigh, factor function, canonical function, Cauchy type integral

1. Denote by Γ some simple closed Jordan line dividing the plane into the domains D_Γ^+ and D_Γ^-. It is assumed that $\infty\in D_\Gamma^-$. We will say that $\Gamma\in R$ if the singular integral.

$$(S¨)(²)=\frac{1}{®i}\int_\Gamma\frac{¨(t)}{t-²}\,dt \tag{1}$$

generates a bounded operator in the Lebesgue space $L_p(\Gamma)$, $p>1$. Such lines are called regular, sometimes Carleson lines and written as $\Gamma\in R$.

In recent years, the operator (1) has been considered in nonstandard Lebesgue classes $L^{p(\cdot)}(\Gamma)$.

It is said that $¨\in L^{p(\cdot)}(\Gamma)$ if

$$I_p(¨)\equiv\int_\Gamma|¨(t)|^{p(t)}|dt|<\infty,$$

where $¨\,(t)$ is a measurable function, $p(t)$ is a continuous real function and $0 < \underline{p} \le p(t) \le \overline{p} < \infty$, where $\underline{p}$ and $\overline{p}$ are constants.

The norm on the above set of functions is defined as follows

$$\|¨\|_{L^{p(\cdot)}} = \inf\left\{ a > 0,\quad I_p\left(\frac{¨}{a}\right) \le 1 \right\}. \tag{2}$$

The space $L^{p(\cdot)}(\Gamma)$ has become interesting for mathematicians dealing with boundary value problems after it was shown that by imposing certain conditions on $p(t)$ we obtain the boundedness of the operator (1) in $L^{p(\cdot)}$.

Of the real function $p(t)$, $t \in \Gamma$, it is required to satisfy the following conditions:

(a) If $\underline{p} = \inf p(t)$, $\overline{p} = \sup p(t)$, then $\underline{p} > 1$, $\overline{p} < \infty$;

(b) $|p(t_1) - p(t_2)| \le \dfrac{const}{\ln \dfrac{1}{|t_1 - t_2|}}$, where $t_1 \in \Gamma,\ t_2 \in \Gamma,\ |t_1 - t_2| \le \dfrac{1}{2}$.

For our consideration we will also need weight functions. We will say that $\mu(t)$ is a weight and write $\mu \in W^{p(\cdot)}(\Gamma)$ if

$$\left\|\mu S \mu^{-1}¨\right\|_{L^{p(\cdot)}(\Gamma)} \le M_p \|¨\|_{L^{p(\cdot)}(\Gamma)}, \qquad M_p = const.$$

Let us consider the power function

$$°(t) = \prod_{k=1}^{n} |t - t_k|^{~_k}, \qquad -\frac{1}{p} < ~_k < \frac{1}{p'}, \quad k = 1, 2, \ldots, n, \quad t_k \in \Gamma.$$

If $p(\cdot) = p = const$, $\Gamma \in R$, $t_k \in \Gamma$, $k = 1, 2, \ldots, n$, then the function $° \in W_p(\Gamma)$ (see e.g. [1, p. 30]). This result was extended to the space $L_{p(\cdot)}(\Gamma)$. It is proved in [2] that if the Jordan line $\Gamma \in R$, the conditions (1) and the relations

$$-\frac{1}{p(t_k)} < ~_k < \frac{1}{p'(t_k)}, \qquad p'(t_k) = \frac{p(t_k)}{p(t_k) - 1}, \qquad t_k \in \Gamma, \quad k = 1, 2, \ldots, n \tag{3}$$

are fulfilled, then we have

$$°(t) : \prod_{k=1}^{n} |t - t_k|^{~_k} \in W_{p(\cdot)(\Gamma)}.$$

2. Let a simple closed curve $\Gamma \in R$ divide the complex plane into two domains D_Γ^+ and D_Γ^-, where $\infty \in D^-$. The direction on Γ, for which the domain D_Γ^+ remains on the left is assumed to be positive.

When considering a boundary value problem, usually the Cauchy type integral is used

$$(K¨)(z) := \frac{1}{2® i} \int_\Gamma \frac{¨(t)}{t - z} dt. \tag{4}$$

We denote by $K_{p(\cdot)}(D_\Gamma^\pm)$ the class of functions $\Phi(z)$ that can be represented by the formula (4) in D_Γ^+ and D_Γ^-, respectively, and assume that $¨ \in L^{p(\cdot)}(\Gamma)$.

Furthermore, if $\Phi(z)=\Phi_0(z)+P(z)$, where $\Phi_0(z)\in K_{p(\cdot)}\left(D_\Gamma^{\pm}\right)$ and $P(z)$ is a polynomial, then as usual we denote the class of such functions by $\tilde{K}_{p(\cdot)}$.

3. We call the following problem a boundary value problem of linear conjugation: Find analytic functions in $D_\Gamma^{\pm}$ which belong to the definite classes and, for $t\in\Gamma$, satisfy the following condition

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t), \tag{5}$$

where G and g are given functions, $m<\left|G(t)\right|<M$, $m\neq 0$, while the sought function $\Phi(z)$ belongs to the preliminarily defined classes.

We will consider the problem formulated as follows: Find a function $\Phi(z)\in K_\Gamma^{p(\cdot)}$ when $G(t)$ is a piecewise continuous function on Γ with discontinuity points $\{t_k\}_{k=1}^n$, $m<\left|G(t)\right|<M$, $m>0$, $g\in L^{p(\cdot)}(\Gamma)$. The line Γ is assumed to be simple and closed and belonging to the class R, and satisfying additional conditions at the points $\{t_k\}_{k=1}^n$.

To formulate these conditions we recall the result due to Seifulaev [3] which we write in the following form: If $\Gamma\in R$ is a closed Jordan curve, $t_0\in\Gamma$, $\arg(z-t_0)$, is some fixed branch on the plane cut along the line $\Gamma_{t_0\infty}\in D_\Gamma^-$ (Γ_{ab} is an open continuous line with ends a and b directed from a to b), then there exist the limits

$$\varlimsup_{\substack{t\to t_0\\ t\in\Gamma}}\frac{\arg(t-t_0)}{\ln|t-t_0|} \quad\text{and}\quad \varliminf_{\substack{t\to t_0\\ t\in\Gamma}}\frac{\arg(t-t_0)}{\ln|t-t_0|}.$$

In the present work, our requirements are somewhat larger, namely: for all discontinuity points of the function $G(t)$, i.e. at the points $\{t_k\}_{k=1}^n$ there exist one-sided limits

$$\lim_{\substack{t\to t_k^+\\ t\in\Gamma}}\frac{\arg(t-t_k)}{\left|\ln|t-t_k|\right|}=\Delta_k^+, \qquad \lim_{\substack{t\to t_k^-\\ t\in\Gamma}}\frac{\arg(t-t_k)}{\left|\ln|t-t_k|\right|}=\Delta_k^- \tag{6}$$

and, besides, for points t_k there exist small arc-wise neighborhoods $\text{¢}_k^+\in\Gamma_{t_{k-1},t_k}$ and $\text{¢}_k^-\in\Gamma_{t_k,t_{k+1}}$ for which we have

$$\arg(t-t_k)=\Delta_k^{\pm}\left|\ln|t-t_k|\right|+O(1), \qquad t\in\text{¢}_k^{\pm}. \tag{7}$$

The final formulation of the problem reads as follows: Find functions $\Phi(z)\in K_\Gamma^{p(\cdot)}\left(D_\Gamma^{\pm}\right)$ satisfying the boundary condition (5) if Γ is a simple closed line of the class R, $G(t)$ is a given piecewise continuous function on Γ, $g(t)\in L^{p(\cdot)}(\Gamma)$ and the line Γ at the discontinuity points of the function $G(t)$ satisfies the condition (7).

If $\Gamma=\Gamma_{ab}$, $G(t)$ is a continuous function on Γ, the condition (7) is fulfilled at the points a and b, $p(\cdot)=p=const$, we solved the problem of linear conjugation in [4]. For $L^{p(\cdot)}$, it is shown in [5] that corresponding singular operator is Noetherian when at the discontinuity points of the function $G(t)$ the conditions on the line Γ are more rigid than (7). The problem was for the first time posed in $L^{p(\,\cdot\,)}$ somewhat earlier in [6] and solved completely when unilateral tangents are given at the discontinuity points of the function $G(t)$.

If $G(t)$ and $g(t)$ are piecewise Hölder and Γ is a smooth line, the problem is formulated and solved in [7], [8] and other papers. These cases are called classical.

4. Let us assume $\{t_k\}_{k=1}^{n}$, $t_k \in \Gamma$, $t_{n+1} = t_1$ are the discontinuous points of function $G(t)$. For the open arc with the ends *a* and *b* we use the notation Γ_{ab}. We consider direction from *a* to *b* as positive. In notation $t \to t_k^{\pm}$ we mean that tending is inside $\Gamma_{t_k t_{k+1}}$, i.e. $t \in \Gamma_{t_k t_{k+1}}$. Also, $t_k \to t_k^{-}$ as $t \in \Gamma_{t_{k-1} t_k}$.

Denote

$$G(t_k+0) \equiv \lim_{t \to t_k^{+}} G(t) \quad \text{and} \quad G(t_k-0) \equiv \lim_{t \to t_k^{-}} G(t).$$

To solve the problem (5) in the formulation given above, following [9], [4] we write the function $G(t)$ in the form

$$G(t) = G_1(t) \cdot G_2(t), \tag{8}$$

where

$$G_1(t) = \exp \mu_1(t), \qquad G_2(t) = \exp \mu_2(t),$$
$$\mu_1(t) + \mu_2(t) = \ln G(t),$$
$$\mu_2(t) = \sum_{k=1}^{n} \left[\ln G(t_k-0) + \frac{\ln G(t_k+0) - \ln G(t_k-0)}{t_{k+1}-t_k}(t-t_k) \right] ¡\,(t_k, t_{k+1}),$$

$¡\,(t_k, t_{k+1})$ is the characteristic function of the set $\left\{t:\ t \in \Gamma_{t_k t_{k+1}}\right\}$.

By analogy with [10], we call the function $X(z)$ the canonical function for $G(t)$ if $X(z) \in \tilde{K}^{p(\,\cdot\,)}\left(D_\Gamma^{\pm}\right)$, $X^{-1}(z) \in K^{(p(\,\cdot\,))'}\left(D_\Gamma^{\pm}\right)$, where

$$\left(p(\,\cdot\,)\right)' = \frac{p(\,\cdot\,)}{p(\,\cdot\,)-1}$$

and

$$X^{+}(t) = G(t)\, X^{-}(t).$$

If in addition $X^{+} \in W^{p(\,\cdot\,)}(\Gamma)$, then $X(z)$ is called a factor function.

Denote

$$X_1(z) := \begin{cases} \prod\limits_{k=1}^{n} (z-t_k)^{\aleph_k} \exp\left(K_\Gamma \ln G_1\right)(z), & z \in D_\Gamma^{+}, \\ (z-z_0)^{\aleph} \left(\dfrac{z-t_k}{z-z_0}\right)^{\aleph} \left(\exp K_\Gamma \ln G_1\right)(z), & z \in D_\Gamma^{-}, \end{cases} \tag{9}$$
$$X_2(z) := \left(\exp K_\Gamma \ln G_2\right)(z).$$

Keeping in mind that the function $\mu_2(t)$ is continuous, we have

Lemma 1. $X_2(z)$ *is the canonical function for* $G_2(t)$ *in the space* $L_p(\Gamma)$ *for* $\forall\ p>1$, $p = const$.

One of the basic assertions used for the consideration of the problem (5) as it is formulated in Subsection 3 is

Lemma 2. *If* $\Gamma \in R$ *, the condition* (7) *is satisfied at the points* $\{t_k\}_{k=1}^n$ *and for* $t \in \text{¢}_k$ *,* $\text{¢}_k \subset \Gamma$ *denotes a small arc-wise neighborhood of the point* $t_k \in \Gamma$ *, then we obtain*

$$(K\mu_1)^+(t) = M_k(t)\left(|t-t_k|\right)^{\aleph_k + \sim_k}$$

where $\copyright_k$ are integer numbers and

$$-\frac{1}{p(t_k)} < \sim_k < \frac{1}{\left(p(t_k)\right)'},$$

$\left(p(t_k)\right)'$ *is the same as in* (3).

Using Lemma 2, we obtain

Lemma 3. *If* $\Gamma \in R$ *and the condition* (7) *is fulfilled at the points* $\{t_k\}_{k=1}^n$*, then there exists* $\pounds > 0$ *such that* $X_1(z) \in K^{p(\cdot)+\pounds}\left(D_\Gamma^+\right)$ *and* $X_1^{-1}(z) \in K^{\left(p(\cdot)+\pounds\right)'}\left(D_\Gamma^-\right)$, $\aleph_k$ *is the same as in Lemma* 1 *and*

$$\aleph = \sum_{k=1}^{n} \aleph_k . \tag{10}$$

To consider the case $z \in D_\Gamma^-$, taking $z_0 \in D_\Gamma^+$ and the transformation $\text{¸} = (z - z_0)^{-1}$, we obtain an analogous assertion for D_Γ^-. Also applying the formula (3) we obtain

Theorem 1. *If* $\Gamma \in R$ *and the condition* (7) *is fulfilled at the points* $\{t_k\}_{k=1}^n$*, then we obtain the function* $X_1(z)$ *is a factor function for* $G_1(t)$ *in* $K^{p(\cdot)+\pounds}\left(D_\Gamma^{\pm}\right)$ *with the index*

$$\aleph = \sum_{k=1}^{n} \aleph_k$$

($\aleph_k$ *is the same as in Lemma* 2).

Theorem 2. *If* $\Gamma \in R$ *and the condition* (7) *is fulfilled at the discontinuity points of the function* $G(t)$ *which are everywhere denoted by* $\{t_k\}_{k=1}^n$*, then the function*

$$X(z) = \prod_{k=1}^{n} (z - t_k)^{-\aleph} \left(\exp K \ln G\right)(z) \tag{11}$$

or, which is the same,

$$X(z)=\begin{cases}\prod_{k=1}^{n}\left(z-t_k\right)^{-\aleph}\left(\exp K\ln G\right)(z), & z\in D_\Gamma^+,\\ \left(z-z_0\right)^{\aleph}\sum_{k=1}^{n}\left(\dfrac{z-t_k}{z-z_0}\right)^{\aleph_k}\left(\exp K\ln G_1\right)(z), & z\in D_\Gamma^-\end{cases} \qquad (12)$$

is the canonical function for $G(t)$ *in* $K^{p(\,\cdot\,)+£}\left(D_\Gamma^\pm\right)$ *with index* $\aleph$ (*as usual the index is the order* $X(z)$ *at infinity*), $\aleph_k$ *is the same as in Lemma* 2 *and* $\aleph=\sum_{k=1}^{n}\aleph_k$.

5. Let us proceed to the solution of the boundary value problem (5) as it is formulated in Subsection 3. As usual we denote

$$P:=\frac{1}{2}(I+S), \quad Q:=\frac{1}{2}(I-S).$$

The problem (1) is equivalent to solving the equation

$$P^{\cdot\cdot}+GQ^{\cdot\cdot}=g$$

in $L^{p(\,\cdot\,)}(\Gamma)$.

Consider the operator

$$A:=P^{\cdot\cdot}+GQ^{\cdot\cdot}\ .$$

Using the method well tested by many authors (see e.g. [11]) we approximate the continuous function $G_2(t)$ by rational functions. This gives us the possibility to show that for $\aleph=0$ the operator A is invertible in $L^{p(\,\cdot\,)}(\Gamma)$. Therefore the canonical function (10) constructed in Subsection 4 will be the factor function for $G(t)$. Further it is easy to show that for the problem (5) the classical results hold true.

Finally, we have

Theorem 3. *If* Γ *is a simple closed line of the class* $R, G(t)$ *is a piecewise-continuous function on* Γ *with discontinuity points* $\{t_k\}_{k=1}^{n}$, $g(t)\in L^{p(\,\cdot\,)}(\Gamma)$, *the line* Γ *satisfies the condition* (7) *at the points* t_k, $k=1,2,\ldots,n$, *then the solution* (*if any*) *of problem* (5) *in* $K^{p(\,\cdot\,)}(\Gamma)$ *has the form*

$$\Phi(z)=X(z)\int_{\Gamma_{ab}}\frac{g(t)}{X^+(t)(t-z)}dt+P_{\aleph-1}(z)X(z), \qquad (13)$$

where $P_n(z)$ is the polynomial of the n-th degree for $n\ge 0$ and $P_n(z)\equiv 0$ if $n<0$.

If $\aleph>0$, then the problem has $\aleph$ linearly independent solutions; if $\aleph=0$, then the solution is unique; if $\aleph<0$, then for the solvability of the problem it is necessary and sufficient that the conditions

$$\int_{\Gamma_{ab}}\frac{t^k g(t)}{X^+(t)}dt=0, \quad k=0,1,\ldots,\aleph-1$$

be fulfilled.

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განხილულია წრფივი შეუღლების სასაზღვრო ამოცანა

$$¤^{+}(t)=G(t)¤^{-}(t)+g(t), \qquad t\in\Gamma$$

სასაზღვრო პირობით, სადაც Γ **მარტივი შეკრული კარლესონის წირია,** $G(t)$ **და** $g(t)$ **მოცემული ფუნქციებია** Γ**-ზე,** $g(t)\in L^{p(\cdot)}(\Gamma)$ **და** $G(t)$ **უბან-უბან უწყვეტია,** $0<m<|G(t)|<M<\infty$**,** $L^{p(\cdot)}(\Gamma)$ **ცვლადმაჩვენებლიანი ლებეგის სივრცეა. საძიებელი ფუნქცია წარმოდგენადი უნდა იყოს კოშის ტიპის ინტეგრალით** $D_{\Gamma}^{\pm}$**-ში.** $G(t)$ **ფუნქციის წყვეტის წერტილებში** Γ **წირზე მოითხოვება გარკვეული პირობა, რომელიც ნაკლები შეზღუდვაა, ვიდრე ეს მოცემული იყო სხვა შრომებში. იწერება ცხადი ამოხსნები.**

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