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Mathematics

On the Origin of the Theory of Distributions or the Heritage of Andrea Razmadze

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(Presented by Academy Member Hvedri Inassaridze)

ABSTRACT. The paper deals with the problem of creating the theory of distributions and is an attempt to show that Andrea Razmadze stood behind the creation of this theory. He was the first to have introduced the class of finite-jump functions, which are considered to be both native solutions (extremals) of some variational problems and foundations for creating the generalized functions. *© 2017 Bull. Georg. Natl. Acad. Sci.*

Key words: Theory of Distributions, class of finite-jump functions, Euler-Lagrange Lemma, extremals

The Theory of Distributions (ThDis), creation of which was first attributed to S.Sobolev [1] and L.Schwartz [2], could be hardly compared with any other achievements in mathematics.

The aim of this article is to show that Andrea Razmadze stood at the origin of creation of ThDis [3-6]. We suppose that he was the first to have introduced and investigated the class of finite-jump functions, which are considered to be both the native solutions (named as extremals) of some variational problems and foundations for creation of the ThDis.

A powerful stimulus for writing this paper were the talks with G.Chogoshvili about the heritage of Razmadze in November, 1989, and in this connection consideration of some parts of the excellent monograph by L.Young where we read: "The Euler-Lagrange Lemma (E.-L.Lemma), which is basic in derivation of Euler's differential equation (3.6), amounts to the first use of what we now call Schwartz distributions" [7: 17].

The forms of E.-L.Lemma in [7] and [5] are the same, but the difference is that the admissible continuous functions with finite-jumping derivatives are in [7] and with continuous derivatives – in [3,5]. We must note that to prove E.-L.Lemma according to [7] its simple form is considered $\lambda(x)=0$ and test functions are stump-shaped ones [7: 18-19]. It is easy to see that the methodology of proving E.-L.Lemma by [3,5] is applicable directly to the class of finite-jump functions. In reality we used the scheme of proof from [3,5] and considered $\varphi(z), z\in(0,1), x<\xi, 0<\varepsilon<\frac{\xi-x}{2}$ defined as:

$$\text{¨}(z) := \{0, 0 < z < x; z - x + 0.1\text{£}, x < z < x + \text{£}; \text{£}, x + \text{£} < z < \text{¶} - \text{£};$$
$$-z + \text{¶} + 0.1\text{£}, \text{¶} - \text{£} < z < \text{¶}; 0, \text{¶} < z < 1\}.$$

Then in our case (1) and the following inequality from [5: 39] will have such forms:

$$\int_{x}^{x+\text{£}} \text{ª}(z)dz - \int_{\text{¶}-\text{£}}^{\text{¶}} \text{ª}(z)dz + \text{£}\int_{x+\text{£}}^{\text{¶}-\text{£}} \text{«}(z)dz = -\int_{x}^{x+\text{£}} \text{«}(z)(z - x + 0.1\text{£})dz - \int_{\text{¶}-\text{£}}^{\text{¶}} \text{«}(z)(-z + \text{¶} + 0.1\text{£})dz, \tag{1}$$

$$\left| \int_{x}^{x+\text{£}} \text{ª}(z)dz - \int_{\text{¶}-\text{£}}^{\text{¶}} \text{ª}(z)dz + \text{£}\int_{x+\text{£}}^{\text{¶}-\text{£}} \text{«}(z)dz \right| \le 2.2G\text{£}^{2},\ G = \max|\text{«}(z)|,\ (0 \le z \le 1). \tag{2}$$

Now, if we repeat the end of proving process of Lemma [5: 40] we will have

$$\text{ª}'(x) = \text{«}(x)$$

for the class of finite-jump functions. This class is wider, than the class of functions considered according to [7].We note here that the same class of functions in [7] was considered by Razmadze in [4,6].

After 1921. the following important step was made by Razmadze [4,6], when he investigated the problem of existence of discontinuous extremals for the functional

$$J = \int_{0}^{1} f\left(x, y(x), y'(x)\right) dx ,$$

and considered the curves of such solution of the above problem having the point of finite discontinuity and represented as limit of a sequence of continuous functions. The latter ones are named as approximate curves in [4].

(We note that Razmadze considered

$$J = \int_{0}^{1} \sin\left[y(x)y'(x)\right] dx,$$

as the class of extremals having every point of finite discontinuity [6,§30:126-128]).

Now it is necessary to remember that the class of ThDis was also introduced and investigated by Polish mathematicians [8] as limit of sequences of continuous functions by applying Cantor's notion of equivalent classes for real numbers. These authors also proved that the ThDis by Sobolev and Schwartz were the same as the sequential approach in [8].

Thus, Razmadze introduced the class of discontinuous extremals for proving E.L.-Lemma in [3,5] and as limit of sequences of continuous functions according to [4,6]. The uniqueness of the solution proves that both classes are identical and are defined as basic ones for ThDis.

In the historical sense, it is evident, that the problems arose after publications of K.Weierstrass and J.Hadamard landmark examples connected with justifying "Dirichlet Principle."The influence of D.Hilbert [9], who generalized the above problem and created the rigorous theory of the basic equations of "Mathematical Physics" is great even today. At the same time D.Hilbert, H.Poincaré, J.Hadamard, S.Bernstein, C.Carathéodory, L.Tonelli, R.Courant, O.Bluementhal and many other famous mathematicians understood that for the variational problems in general the Weierstrass example was essential according to L.Young [7]. Thus the principal problem of justification of "The Variational Calculus" was open. Razmadze considered the problem, containing the Weierstrass example too, which not only solved the essential problem of an important branch of mathematics, but he introduced and considered the class of finite-jump discontinuous functions as original and basic for ThDis.

მათემატიკა

განზოგადებული ფუნქციის შექმნის საწყისებთან, ანუ ანდრია რაზმაძის მემკვიდრეობის შესახებ

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(წარმოდგენილია აკადემიის წევრის ხ. ინასარიძის მიერ)

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