საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 11, №2, 2017
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 11, no. 2, 2017

Cybernetics

Probabilistic Analysis of a Redundant Repairable System with Two Maintenance Operations

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ABSTRACT. A redundant system with two types of maintenance services, single repair server and single replacement server is considered. Repair time and replacement time are random variables. Mathematical model for dependability and performance analysis is constructed and investigated. Explicit solution of steady-state probabilities is obtained in terms of operational calculus (Laplace transformations).
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Key words: probabilistic analysis, redundant system, repair, replacement

In the classical mathematical theory of reliability, particularly in redundancy theory, productive methods and models for redundant systems analysis and optimization are developed and successfully applied in all the stages of such systems' life cycle [1].

At the same time the study of redundant systems with repairable units is mainly focused on the analysis of their lifetime and the probabilistic characteristics of their stay in various states of serviceability [2,3].

As for the downtime (non-serviceable state) of redundant systems, little attention was paid to. The downtime by itself often coincides with duration of replacement time of a failed main unit by the redundant one. The fact is that the replacement of the failed unit was quite regarded as instantaneous (its duration was so small that it was acceptable to be ignored), and the repair time of failed unit was significantly small compared to its lifetime. In these conditions, the mean duration of downtime is really negligible [4].

This approach contributed to the fact that in most studies of the reliability of redundant systems (except for some simple cases), the replacement of the failed unit was not considered as a separate, independent maintenance operation.

On the other hand, in modern conditions, this assumption is far from the reality [3-5].

As a matter of fact, nowadays it is important not only to estimate the mean downtime, but also the nature of its dispersion around mean value. Therefore, the problem of a full probabilistic analysis of downtime (including replacement time) is very urgent.

Simultaneously, duplex and triplex systems are most widespread in redundancy practice. In the presented paper we investigate one of such type triplex systems.

Subject of Study and its Mathematical Description

The redundant technical system consists of one main and two redundant units. The failure of a main unit happens with intensity α and that of redundant ones with the intensity β, $0 < \beta \le \alpha$. The failed main unit is replaced by operating redundant one after the replacement of the latter. While failed elements, both main and redundant one, are transmitted for repair, we have one repair server and one replacement server. Suppose that repair time of the failed unit is a random variable with exponential distribution with parameter μ. Replacement time distribution function is arbitrary H. The repaired unit renews all its initial qualities and is included in the group of the redundant units. Denote replacement rate by $\lambda(u)$ so that $\lambda(u) = \frac{h(u)}{1 - H(u)}$, where $h(u) = H'(u)$.

We introduce the random processes, which define the states of the considered system at the time instant t:

$i(t)$ is the number of units missing in the group of main units;

$j(t)$ is the number of non-operative (failed) units in the system;

$\theta(t)$ is the time interval length from the beginning of the replacement operation to the time instant t.

In order to describe the system state we define probability characteristics:

$P(j,t) = P\{i(t) = 0,\ j(t) = j\},\ \ j = 0,1,2.$

$R(1,t) = P\{i(t) = 1,\ j(t) = 3\},$

$q(i,j,t,u) = \lim_{h \to 0} P\left(\frac{1}{h} P\{i(i) = i, j(t) = j - 1,\ u < \theta(t) < u + h\}\right) \quad i = 1,\ j = 1,2,3$

$P(j,t)$, $R(i,t)$ and $q(i,j,t,u)$ have great theoretical and practical values, as they allow us easily express other characteristics of the system.

Suppose, functions $P(j,t)$ and $R(1,t)$ have continuous derivatives, when $t > 0$ and functions $q(i,j,t,u)$ have continuous partial derivatives when $t > 0$, $u \ge 0$.

According to the usual probabilistic considerations the following theorems can be proved.

Theorem 1. *In the mentioned conditions functions* $P(j,t)$ *and* $R(1,t)$ *satisfy the following system of integro-differential equations:*

$$\frac{dP(0,t)}{dt} = -(\alpha + 2\beta)P(0,t) + \mu P(1,t) + \int_0^t q(1,1,t,u)\lambda(u)\,du;$$

$$\frac{dP(1,t)}{dt} = -(\alpha + \beta + \mu)P(1,t) + 2\beta P(0,t) + \mu P(2,t) + \int_0^t q(1,2,t,u)\lambda(u)\,du; \tag{1}$$

$$\frac{dP(2,t)}{dt} = -(\alpha+\mu)P(2,t) + \beta P(1,t) + \int_0^t q(1,3,t,u)\lambda(u)\,du;$$

$$\frac{dR(1,t)}{dt} = -\mu R(1,t) + \alpha P(2,t).$$

With initial conditions

$$P(0,0)=1,\ P(1,0)=P(2,0)=R(1,0)=0\,.$$

Theorem 2. *Functions* $q(i,j,t,u)$ *satisfy the following system of partial differential equations:*

$$\frac{\partial q(1,1,t,u)}{\partial t} + \frac{\partial q(1,1,t,u)}{\partial u} = -(2\beta+\lambda(u))q(1,1,t,u) + \mu q(1,2,t,u);$$

$$\frac{\partial q(1,2,t,u)}{\partial t} + \frac{\partial q(1,2,t,u)}{\partial u} = -(\beta+\mu+\lambda(u))q(1,2,t,u) + \mu q(1,3,t,u) + 2\beta q(1,1,t,u); \quad (2)$$

$$\frac{\partial q(1,3,t,u)}{\partial t} + \frac{\partial q(1,3,t,u)}{\partial u} = -(\mu+\lambda(u))q(1,3,t,u) + \beta q(1,2,t,u).$$

With boundary conditions

$$\begin{aligned} q(1,1,t,0) &= 0,\\ q(1,2,t,0) &= \alpha P(0,t),\\ q(1,3,t,0) &= \alpha P(1,t) + \mu R(1,t). \end{aligned} \quad (3)$$

For further study of the system we will consider the event $A_{ij}(u)$*, which is defined as follows.*

$$\left\{\begin{array}{c}\text{The number of failed elements at the end of the segment with length u is j}\\ \text{...if the number of failed element at the beginning of the segment is i}\end{array}\right\}$$

Further, assume that $P_{ij}(u) = P\{A_{ij}(u)\}$.

According to the usual probabilistic considerations we obtain the expression for $q(i,j,t,u)$:

$$\begin{aligned} q(1,1,t,u) &= q(1,2,t-u,0)\overline{H(u)}P_{10}(u) + q(1,3,t-u,0)\overline{H(u)}P_{20}(u)\\ q(1,2,t,u) &= q(1,2,t-u,0)\overline{H(u)}P_{11}(u) + q(1,3,t-u,0)\overline{H(u)}P_{21}(u)\\ q(1,3,t,u) &= q(1,3,t-u,0)\overline{H(u)}P_{22}(u) + q(1,2,t-u,0)\overline{H(u)}P_{12}(u), \end{aligned} \quad (4)$$

where $\overline{H(u)} = 1 - H(u)$.

Probabilities $P_{ij}(u)$ are received from the following systems:

$$\begin{cases} P_{00}(u) = e^{-2\beta u} + 2\int_0^u \beta e^{-2\beta v} P_{10}(u-v)\,dv\\ P_{10}(u) = \int_0^u \mu e^{-\mu v} e^{-\beta v} P_{00}(u-v)\,dv + \int_0^u \beta e^{-\beta v} e^{-\mu v} P_{20}(u-v)\,dv\\ P_{20}(u) = \int_0^u \mu e^{-\mu v} P_{10}(u-v)\,dv \end{cases} \quad (5)$$

and

$$\begin{cases} P_{22}(u) = e^{-\mu u} + \int\limits_0^u \mu e^{-\mu v} P_{12}(u-v)\,dv \\ P_{12}(u) = \int\limits_0^u \mu e^{-\mu v} e^{-\beta v} P_{02}(u-v)\,dv + \int\limits_0^u \beta e^{-\beta v} e^{-\mu v} P_{22}(u-v)\,dv \\ P_{02}(u) = 2\int\limits_0^u \beta e^{-2\beta v} P_{12}(u-v)\,dv \end{cases} \tag{6}$$

To verify equity of the second relation in the system (5) we use the following considerations.

The event $A_{10}(u)$ can be represented by the sum of two mutually exclusive events:

1. At the beginning of the time interval $[0,u]$ there was one failed element, which was repaired at the time instant v. From at moment, there has been an event $A_{00}(u-v)$. Taking into consideration all the possible values of v the probability of this event is the first term on the left side.

2. At the beginning of the time interval there were one failed and one operative units. The operative element has failed at the time instant v. From this moment, there was an event $A_{20}(u-v)$. Taking into consideration all possible values of v the probability of this event is the second term on the left side.

By adding this probabilities the second equation in (5) wos obtained. Likewise proved the equity of the other equations in (5) and (6) are. Equations in (7) are received from the normalization condition.

From those systems it is true to express values for $P_{ij}(u)$:

$$P_{ij}(u) = a_{ij} + b_{ij}e^{-\gamma u} + c_{ij}e^{-\delta u}, \tag{8}$$

Where $\gamma = \dfrac{3\beta + 2\mu + \sqrt{\beta^2 + 4\beta\mu}}{2}$ and $\delta = \dfrac{3\beta + 2\mu - \sqrt{\beta^2 + 4\beta\mu}}{2}$,

$$a_{20} = \frac{\mu^2}{\Delta_a},\ b_{20} = \frac{\mu^2}{\Delta_b},\ c_{20} = \frac{\mu^2}{\Delta_c},\ a_{00} = \frac{\mu^2}{\Delta_a},\ b_{00} = \frac{\beta^2\left(1+\sqrt{\beta+4\mu}\right)}{\Delta_b},\ c_{00} = \frac{\beta^2\left(1\sqrt{\beta+4\mu}\right)}{\Delta_c}$$

$$a_{10} = \frac{\mu^2}{\Delta_a},\ b_{10} = -\frac{\beta\mu\left(3-\sqrt{\beta+4\mu}\right)}{\Delta_b},\ c_{10} = -\frac{\beta\mu\left(3+\sqrt{\beta+4\mu}\right)}{\Delta_c},$$

$$a_{22} = \frac{2\beta^2}{\Delta_a},\ b_{22} = -\frac{\beta\mu\left(1-\sqrt{\beta+4\mu}\right)}{2\Delta_b},\ c_{22} = -\frac{\beta\mu\left(1+\sqrt{\beta+4\mu}\right)}{2\Delta_c},$$

$$a_{12} = \frac{2\beta^2}{\Delta_a},\ b_{12} = \frac{\beta^2\left(1-\sqrt{\beta+4\mu}\right)-2\beta\mu}{2\Delta_b},\ c_{12} = \frac{\beta^2\left(1+\sqrt{\beta+4\mu}\right)-2\beta\mu}{2\Delta_c}$$

$$a_{21} = \frac{2\beta\mu}{\Delta_a},\ b_{21} = -\frac{2\mu^2-\beta\mu\left(1-\sqrt{\beta+4\mu}\right)}{2\Delta_b},\ c_{21} = -\frac{2\mu^2-\beta\mu\left(1+\sqrt{\beta+4\mu}\right)}{2\Delta_c},$$

$$a_{11}=\frac{2\beta\mu}{\Delta_a},\ b_{11}=\frac{\beta\mu\left(5-\sqrt{\beta+4\mu}\right)-\beta^2\left(1+\sqrt{\beta+4\mu}\right)}{2\Delta_b},\ c_{11}=\frac{\beta\mu\left(5+\sqrt{\beta+4\mu}\right)-\beta^2\left(1-\sqrt{\beta+4\mu}\right)}{2\Delta_c},$$

And $\Delta_a=\mu^2+2\beta\mu+2\beta^2$, $\Delta_b=\dfrac{\beta^2+4\beta\mu+(3\beta+2\mu)\sqrt{\beta^2+4\beta\mu}}{2}$,

$$\Delta_c=\frac{\beta^2+4\beta\mu-(3\beta+2\mu)\sqrt{\beta^2+4\beta\mu}}{2}.$$

Consider the system in steady-state conditions and introduce the notations:

$$P(j)=\lim_{t\to\infty}P(j,t),\quad j=\overline{0,2};\ R(1)=\lim_{t\to\infty}R(1,t);\ q(1,j,u)=\lim_{t\to\infty}q(1,j,t,u),\ j=\overline{1,3}.$$

From (5) we get

$$(\alpha+2\beta)P(0)=\mu P(1)+\int_0^\infty q(1,1,u)\lambda(u)du;$$

$$(\alpha+\beta+\mu)P(1)=2\beta P(0)+\mu P(2)+\int_0^\infty q(1,2,u)\lambda(u)du; \tag{9}$$

$$(\alpha+\mu)P(2)=\beta P(1)+\int_0^\infty q(1,3,u)\lambda(u)du;$$

$$\mu R(1)=\alpha P(2).$$

And for boundary conditions:

$$q(1,1,0)=0, \tag{10}$$

$$q(1,2,0)=\alpha P(0),$$

$$q(1,3,0)=\alpha P(1)+\mu R(1).$$

Taking into account expressions (8) and boundary conditons (10) in (4) we obtain the following steady-state results:

$$q(1,1,u)=\alpha P(0)\overline{H(u)}\left(a_{10j}+b_{10}e^{-\gamma u}+c_{10}e^{-\delta u}\right)+$$

$$+\left(\alpha P(1)+\mu R(1)\right)\overline{H(u)}\left(a_{20}+b_{20}e^{-\gamma u}+c_{20}e^{-\delta u}\right)$$

$$q(1,2,u)=\alpha P(0)\overline{H(u)}\left(a_{11}+b_{11}e^{-\gamma u}+c_{11}e^{-\delta u}\right)+$$

$$+\left(\alpha P(1)+\mu R(1)\right)\overline{H(u)}\left(a_{21}+b_{21}e^{-\gamma u}+c_{12}e^{-\delta u}\right) \tag{11}$$

$$q(1,3,u)=\alpha P(0)\overline{H(u)}\left(a_{12}+b_{12}e^{-\gamma u}+c_{12}e^{-\delta u}\right)+$$

$$+\left(\alpha P(1)+\mu R(1)\right)\overline{H(u)}\left(a_{22}+b_{22}e^{-\gamma u}+c_{22}e^{-\delta u}\right).$$

To find the unknown values $P(0)$, $P(1)$, $P(2)$ and $R(1)$ we put (11) expressions into (9) and we obtain the system of algebraic equations relatively to these values:

$$P(0)\left[\alpha\left(1-a_{10}-b_{10}\bar{h}(\gamma)-c_{10}\bar{h}(\delta)\right)\right]-P(1)\left[\mu+\alpha\left(a_{20}+b_{20}\bar{h}(\gamma)+c_{10}\bar{h}(\delta)\right)\right]-$$
$$-R(1)\mu\left[a_{20}+b_{20}\bar{h}(\gamma)+c_{20}\bar{h}(\delta)\right]=0 \quad (12)$$
$$P(0)\left[2\beta+\alpha\left(a_{11}+b_{11}\bar{h}(\gamma)+c_{10}\bar{h}(\delta)\right)\right]-P(1)\left[\mu+\beta+\alpha\left(1-a_{21}-b_{21}\bar{h}(\gamma)-c_{21}\bar{h}(\delta)\right)\right]+$$
$$+\mu P(2)-R(1)\mu\left[a_{21}+b_{21}\bar{h}(\gamma)+c_{21}\bar{h}(\delta)\right]=0$$
$$P(0)\alpha\left(a_{12}+b_{12}\bar{h}(\gamma)+c_{12}\bar{h}(\delta)\right)+P(1)\left[\beta+\alpha\left(a_{22}+b_{22}\bar{h}(\gamma)+c_{22}\bar{h}(\delta)\right)\right]-$$
$$+P(2)[\alpha+\mu]+R(1)\mu\left[a_{22}+b_{22}\bar{h}(\gamma)+c_{22}\bar{h}(\delta)\right]=0$$
$$\alpha P(2)-\mu R(1)=0.$$

Where $\bar{h}(s)=\int_0^\infty e^{-su}h(u)du.$

To solve the system we replace one equation of the system by normalization condition, which looks as:

$$P(0)+P(1)+P(2)+R(1)+\left[\alpha P(0)+\alpha P(1)+\mu R(1)\right]E\xi=1.$$

Here $E\xi=\int_0^\infty\left(1-H(u)\right)du$ is the mean replacement time of the unit.

Conclusion

In this paper we present and research a mathematical model of redundant technical system with exponentially distributed repair time and arbitrary distributed replacement time. Using probabilistic considerations and Laplace transform technique we obtain explicit solution for steady-state probabilities.

From the results of thie research we can easily derive other important probabilistic characteristics of the system. Direction for future research is the investigation of this system behavior for different distribution functions of the replacement time.

Acknowledgement. The designated Project was fulfilled with financial support of the Shota Rustaveli National Science Foundation - SRNSF (Grant # FR/312/4-150/14). Any idea in this publication is possessed by the authors and may not represent the opinion of the SRNSF.

კიბერნეტიკა

აღდგენადი დარეზერვებული სისტემის ალბათური ანალიზი ორი ტიპის მომსახურებით

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ნაშრომში განხილულია აღდგენადი დარეზერვებული ტექნიკური სისტემა ორი ტიპის მომსახურებით. სისტემაში მოქმედებენ ჩანაცვლების ერთი ორგანო და აღდგენის ერთი ორგანო. ჩანაცვლების და აღდგენის ხანგრძლივობები შემთხვევითი სიდიდეებია. აგებულია სისტემის ანალიზური მოდელი. ოპერაციული აღრიცხვის ტერმინებში (ლაპლასის გარდაქმნები) ცხადი სახით მიღებულია სისტემის ალბათური მახასიათებლები დამყარებულ მდგომარეობაში.

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Received March, 2017