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Mathematics

Discrete Invariants of Quadratic Endomorphisms

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ABSTRACT. We present a number of results concerned with discrete topological and geometric invariants of quadratic endomorphisms of real vector spaces. The main attention is given to the mapping degree and cardinalities of fibres of proper endomorphism. Concerning topological invariants, we present a general estimate for the mapping degree and verify its exactness in various classes. Next, we introduce several geometric invariants derived from the structure of the singular set and discriminant of endomorphism. For stable quadratic endomorphisms, we also consider an additional invariant arising from the classification of stable singularities. Several related results and conjectures are also discussed.

Key words: quadratic form, quadratic mapping, index of vector field, stable mapping, signature of quadratic form

Quadratic mappings between real vector spaces appear in a wide variety of problems and were studied by many authors [1-3]. In many problems it is important to have effectively computable invariants of such mappings with respect to various equivalence relations. The two most natural and important equivalence relations in this context are the topological equivalence and geometric equivalence. In some cases it appears possible to describe the equivalence classes of quadratic mappings using certain discrete invariants like the mapping degree or Euler characteristics of fibres [1, 4]. This problem becomes especially interesting and important in the class of mappings which are stable with respect to one of the aforementioned equivalence relations. Recall that a map F is called stable with respect to an equivalence relation T if it has a neighborhood consisting of maps T-equivalent to F. A natural general problem is to describe geometric equivalence classes of mappings. In this context, in addition to topological invariants it is natural to consider discrete invariants of the singular set and discriminant of mapping [5].

In this paper we discuss some discrete invariants in the case of quadratic endomorphism of real vector space. One of the most natural and practically important topological invariants is the mapping degree and many of our considerations are concerned with this invariant. Our first main result gives an exact estimate for the mapping degree of quadratic endomorphism (Theorem 2). We also give explicit examples of quadratic endomorphisms with the maximal possible index in all dimensions. These results are obtained using the algebraic formula for the mapping degree developed in [6, 7]. After that we introduce more subtle discrete geometric invariants of endomorphism (the type and Ehresmann graph) and compute them in several cases. Our considerations are partially motivated by some recent results presented in [2, 3]. In conclusion we present several related results and conjectures.

We begin with recalling certain general concepts and results of topology and singularity theory. As usual a continuous mapping F of topological spaces is called *proper* if, for any compact subset Y of the target space, the preimage $F^{-1}(Y)$ is also compact. If F: $\mathbf{R^n} \to \mathbf{R^n}$ is a proper mapping (endomorphism) then an important topological invariant of F is given by the *mapping degree* Deg F defined as follows. From the properness if follows that F can be extended to a continuous self-mapping [F]: $S^n \to S^n$ of the n-sphere S^n obtained as the one-point compactification of R^n. The mapping degree Deg F is defined as the usual topological degree of [F], which is defined for any continuous self-mapping of an oriented n-manifold without boundary [4]. As is well known, Deg F is invariant with respect to proper homotopies and topological equivalence of maps [4]. For differentiable endomorphism F, one can consider its singular set S(F), defined as the set of all points p such that jacobian $J_F(p)$ vanishes, and the set of singular values defined as $\Delta(F)$= F(S(F)). For convenience and brevity we will say that S(F) is the *spoiler* of F and $\Delta(F)$ is the *discriminant* of F. The set of regular values Reg F is defined as the complement of the discriminant $\Delta(F)$. Thus jacobian J_F of F is non-zero at any preimage of a point $p \in$ Reg F.

If a proper endomorphism F as above is algebraic then preimage of any point is finite and Deg F coincides with the sum of the signs of the Jacobian J_F taken over the preimage of any regular value of F [4]. Denote by r(F) the number of components r(F) of the set or regular values Reg F. If F is a proper algebraic endomorphism, then r(F) is finite and the cardinality of the fibre $F^{-1}(q)$ is the same for all points q in a fixed component of Reg F, which follows from the Ehresmann theorem on the fibres of smooth submersion [5]. Thus we obtain a finite list $\tau(F)$ of (not more than r(F)) non-negative integers given by all possible cardinalities of preimages of regular values. The list $\tau(F)$ is called the *type* of F (cf. [2]) (sometimes it is also called the *generic type* of F).

Many of our considerations are connected with the concept of geometric equivalence which is the same as the right-left-equivalence of mappings used in singularity theory [5]. Recall that two differentiable endomorphisms F_1, F_2 are called geometrically equivalent (or RL-equivalent) if there exist two diffeomorphisms S and T of $\mathbf{R^n}$ such that $F_1 = S F_2 T^{-1}$. Obviously, Deg F, r(F) and $\tau(F)$ are the same for geometrically equivalent endomorphisms and they provide typical examples of discrete invariants we are interested in.

In fact, the diffeomorphic type of the spoiler S(F) and discriminant $\Delta(F)$ are also invariants of geometric equivalence but they are often too complicated to work with and so we restrict ourselves to the discrete invariants introduced above.

The concept of geometric stability, in the sense explained in the introduction, plays fundamental role in singularity theory and nonlinear analysis. For a stable endomorphism F, one can introduce a more refined discrete invariant by considering the canonical stratifications $\Omega_S(F)$ and $\Omega_\Delta(F)$ of S(F) and $\Delta(F)$ respectively (see, e.g., [5]). By a stratified version of Ehresmann theorem, for a stable endomorphism, the number of preimages is the same on any stratum [5]. So one may enrich $\tau(F)$ by joining all possible cardinalities of fibres over each stratum of $\Omega_\Delta(F)$. The resulting set of non-negative integers will be denoted $\tau_\Omega(F)$ and called the *stratified type* of F.

Finally, for better understanding of qualitative behaviour of fibres it is useful to describe the changes in topology of fibres when the image point transversally crosses one of the regular strata of discriminant. They can be conveniently represented in the form of the so-called *bifurcation diagram* B(F) which is defined as a graph with vertices labeled by the components of regular values and edges between each pair of adjacent components. In an analogous way one can define the stratified bifurcation diagram $B_\Omega(F)$ having as vertices all strata of the canonical stratification $\Omega_\Delta(F)$. As an illustration, in the sequel we compute these invariants in a few examples.

To give a more clear idea of the results we are aiming at, we discuss first proper quadratic endomorphisms of the plane and present a bunch of results available in this case (Theorem 1), which may be considered as a paradigm for further developments. Similar results are available for homogeneous quadratic endomorphisms in arbitrary dimension (Theorem 2). Recall that a polynomial endomorphism is called *d-homogeneous* if all of its components are homogeneous d-forms. For d=2, we get the definition of homogeneous quadratic endomorphism. Obviously, the zero-sets of its components consist of a system of lines through the origin which for convenience will be called *zero lines*. We are now in a position to formulate and explain the first main result of this note.

Theorem 1. Homogeneous quadratic endomorphism of the plane is proper if and only if the zero-sets of its components have no common zero lines, i.e. $Z_1 \cap Z_2 = \{0\}$. The mapping degree Deg F of a proper quadratic endomorphism can only take values -2, 0, 2. The image of F is convex and the singular set consists of lines the number of which does not exceed two. The singular point at the origin has multiplicity four. The number of points in any regular fibre is even. The maximal number of points in any fibre of F is four. Preimages of points in the bifurcation diagram may consist of one or three points. Two proper quadratic endomorphisms are geometrically equivalent if and only if their bifurcation diagrams are isomorphic as graphs.

Outline of the proof of Theorem 1. The first statement follows from the Hadamard theorem [4]. The second one follows from the combinatorial formula for the degree of homogeneous endomorphism of the plane in terms of the zero-lines of its components. Indeed, it is known that the topological degree of such an endomorphism is equal to one half of the number of interlacing polar lines from Z_1 and Z_2 (see, e.g., [4]). It follows that the absolute value of Deg F cannot exceed two since the number of interlacings cannot exceed four. Convexity of the image is a special case of a general theorem (see, e.g., [1]). The structure of singular set follows from the homogeneity of the Jacobian of F. The fact that the mapping degree is even follows from the fact that the singular point at the origin has multiplicity four. The number of points in a regular fibre has the same parity as the mapping degree. This number does not exceed four by Bezout theorem. It is known that the cardinality of fibre is maximal for a regular point. Over a point of bifurcation diagram, one or two pairs of preimages coincide so the number of preimages can be one or three. The last statement follows by analyzing the list of normal forms given in [3]. These observations can easily be turned into a rigorous proof, which we omit for the reason of space.

Remark 1. Thus, in this case we know the possible structure of all fibres. In particular, possible cardinalities of preimages are 0, 1, 2, 3, 4. The bifurcation diagram may contain up to 5 vertices corresponding to the possible cardinalities of preimages. Notice that here the number of preimages coincides with the Euler characteristic of the fibre. It is easy to calculate it in concrete cases using an algebraic formula for the Euler characteristic developed in [8]. The same formula enables one to effectively verify the non-degeneracy condition. Indeed, non-degeneracy means that the system of three equations $\{f_1(x,y)= 0,\ f_2(x,y) = 0,\ x^2 + y^2 - 1 = 0\}$ does not have real solutions, i.e., the Euler characteristic of this set should vanish.

Remark 2. In a similar way one can obtain a criterion of properness in more general case of non-necessarily homogeneous quadratic endomorphism.

Remark 3. In fact, all statements of Theorem 1 can be proven using case-by-case analysis based on the complete list of geometric equivalence classes of quadratic endomorphisms of the plane given in [3]. In the sequel we compute discrete invariants for some of the normal forms given in [3].

Remark 4. In the situation of Theorem 1 one may also use the concept of local algebra defined as the factor-algebra of the algebra of formal power series over the ideal generated by the components of endomorphism [5]. It is well known that a d-homogeneous endomorphism F of Euclidean space $\mathbf{R}^n$ is proper if and only if its local algebra at the origin $A_0(F)$ has finite dimension called the *multiplicity* of F at the origin. In such cases the multiplicity $m_0(F)$ is always equal to d^n. In other words, the local algebra at the origin [5] is a d^n-dimensional associative algebra. It is natural to wonder how many different isomorphism classes of d^n-dimensional algebras arise in this way. In general this is a difficult and widely open problem but for homogeneous quadratic endomorphism of the plane the answer is simple and easy to prove: there are exactly three isomorphism classes of local algebras and they are classified by the value of local mapping degree at the origin (as was stated above the latter can only equal -2, 0, 2). The local algebra is an invariant of geometric equivalence [5] and can be used for obtaining discrete invariants.

Remark 5. Homogeneous endomorphisms of the plane are not geometrically stable. This follows, in particular, from the fact that the multiplicity of singular point at the origin is bigger than three which is the top (maximal possible) multiplicity of a stable map in this case (multiplicity of cusp). So one may wish to consider their stable perturbations introduced by H.Whitney [4]. In the sequel we give examples of stable perturbations of homogeneous quadratic endomorphisms.

Remark 6. We will show that, in contrast to Theorem 1, the mapping degree is not a complete invariant of geometric equivalence in all dimensions bigger than two. At the same time the fibre type and bifurcation diagram in certain cases enable one to distinguish equivalence classes, which may be considered as a motivation for their investigation in our context.

Remark 7. Theorem 1 can be elaborated for an endomorphism F which is the gradient of a binary cubic form f (homogeneous cubic polynomial in two variables). In this case, an explicit criterion of properness of F is expressed by vanishing of discriminant of f. Using Ehresmann theorem one can show that mapping properties of F do not change in each component of discriminant of f in the space of its coefficients. Investigating these properties for a representative of each component one can obtain a complete list of possible combinations of mapping properties for proper gradients of binary cubic forms. Another way to do the same is to establish which of the normal forms given in [3] correspond to gradient endomorphisms and then perform case-by-case analysis.

We proceed by presenting a remark on a problem of nonlinear analysis concerned with relations between properness and surjectivity. It is well known that in some situations one can prove that a non-proper map cannot be surjective. For example, this trivially holds true for homogeneous endomorphisms of the line. The same is true for homogeneous quadratic endomorphisms of the plane, which can easily be proven using case-by-case analysis of the normal forms given in [3]. However, the implication “non-proper implies non-surjective” is not true in dimensions bigger than three, even for quadratic endomorphisms [1]. At the same time, this implication holds true for gradients of ternary cubic forms.

Proposition 1. Let F be the gradient of a ternary cubic form f. Then if F is non-surjective then F is not proper.

The proof is obtained using case-by-case analysis based on the study of concrete representatives in each component of complement of discriminant in the space of coefficients. Virtual extrapolation of the same strategy gives good evidence that the implication “non-proper implies non-surjective” holds true in the class of all homogeneous quadratic endomorphisms of $\mathbf{R}^3$**.** Similar problems for d-homogeneous endomorphisms with any d>2 are practically non-explored. One can also try to obtain similar results for quasihomogeneous endomorphisms of the plane. More precisely, for which triples $(w_1, w_2; d)$ of quasihomogeneous data each non-proper qh-endomorphism of type $(w_1, w_2; d)$ is non-proper?

We now present an exact upper estimate for the topological degree of a proper quadratic endomorphism in arbitrary dimension n. Recall that the local algebra $A_0(F)$ of non-degenerate homogeneous polynomial endomorphism F is finite-dimensional and has a basis consisting of the classes of monomials. For any polynomial P, by [P] is denoted its class in the local algebra. The number of basis monomials in $A_0(F)$ having a given degree can be found from the Poincare series of F, which is quite simple in case of quadratic endomorphism [5]. The structure of local algebras was thoroughly investigated by many authors. In particular, the so-called Grothendieck duality implies that on any local algebra there exists a non-degenerate invariant quadratic form called Gorenstein form. As was shown in [6, 7] the signature of Gorenstein form is equal to the mapping degree of F [6, 7]. Below we use several further properties of Gorenstein form which can be found in [5-7]. We are now able to give the desired estimate.

Theorem 2. The mapping degree of proper quadratic endomorphism of odd-dimensional real vector space equals zero. The absolute value of mapping degree of proper homogeneous quadratic endomorphism of $\mathbf{R}^{2k}$ does not exceed

$$D(2k,2)=\frac{(3k-1)!}{k!(2k-1)!}$$

This estimate is exact. For any integer p of the same parity and smaller absolute value, there exists a proper homogeneous quadratic endomorphism with the topological degree equal to p. The cardinality of fibre can be any number in the interval $[0, 2^{2k}]$.

Outline of the proof of Theorem 2. The proof is based on the aforementioned result stating that the (local) mapping degree is equal to the signature of the Gorenstein quadratic form A(F) on the local algebra of F. The maximal degree of monomials in a monomial basis of A(F) is n. From the construction of Gorenstein quadratic form follows that the mapping degree is not bigger than the number of elements of monomial basis of the middle degree [7]. If n is odd there are no such monomials so the degree must vanish. For n=2k, the estimate follows from the well known formula for the number of monomials of fixed degree [5]. Exactness follows by consideration of concrete examples. For n=4, put

$$f_1 = 3x_1^2 - 6x_1x_2 - 6x_1x_3 - 6x_1x_4,\ f_2 = - 6x_1x_2 + 3x_2^2 - 6x_2x_3 - 6x_2x_4,$$
$$f_3 = -6x_1x_3 - 6x_2x_3 + 3x_3^2 - 6x_3x_4,\ f_4 = -6\, x_1x_4 - 6x_2x_4 - 6\, x_3x_4 + 3x_4^2,$$

and consider endomorphism $F=(f_1, f_2, f_3, f_4)$. From the general formula for Poincare series of $A_0(F)$ follows that the number of basis monomials of the middle degree coincides with the estimate given in Theorem 2. Computing the multiplication table of this algebra one finds out that the squares of basis monomials of middle degree are equal to positive multiples of the jacobian class [j(F)] in the local algebra. Since the Gorenstein form is obtained as the multiplication in $A_0(F)$ followed by projection on the class of jacobian [j(F)], its signature equals to the number of monomials of middle degree given in the above estimate. By the signature formula the mapping degree is also equal to that estimate. This proves the exactness of the above estimate for n=4. Analogous examples in any even dimension n=2k are given by the formulas:

$$f_i = (k - 0.5)\, x_i\, [(k-0.5)x_i - \sum_{j=1}^{2k} x_j\ ],\ i = 1, \dots, 2k.$$

For each endomorphism of such kind, it is possible to explicitly construct a monomial basis for the local algebra and verify that the signature of Gorenstein form is equal to the number of basis monomials of middle degree. This means that the above estimate is exact in all dimensions. We omit computational details for the reason of space.

Remark 8. Using an analog of Rouchet theorem for mapping degree it is easy to show that the same estimate is valid (and automatically exact) in the class of all proper quadratic endomorphisms.

Remark 9. It follows that, unlike the case where n=2, the mapping degree does not distinguish geometric isomorphism classes. Indeed, it is easy to verify that, for arbitrary n, there exist endomorphisms which are not geometrically equivalent but have the same mapping degree.

Consider, for example,

$$G_n = (x_1^2, \dots , x_n^2), H_n = (2x_1x_2 + x_n^2, 2x_2x_3 + x_1^2, \dots , 2x_nx_1 + x_{n-1}^2)$$

which are the gradient mappings of cubic polynomials $x_1^3 + \dots + x_n^3$ and $x_1^2x_2 + \dots + x_n^2x_1$, respectively. It is easy to see that the local algebras of G_n and H_n are non-isomorphic. So by Mather-Yau theorem [5] these two endomorphisms are not geometrically equivalent. At the same time, if n is odd they both have vanishing topological degree.

This example shows that for geometric classification of endomorphisms in dimensions higher than two one needs more powerful invariants than the mapping degree, which motivates consideration of discrete invariants like the fibre type and bifurcation diagram. In the latter example it is straightforward to see that the two fibre types do not coincide, so the fibre type is really a useful invariant in geometric classification problems.

Remark 10. An endomorphism of vector space can be interpreted as a vector field on this space. In particular, the mapping degree of proper endomorphism is the same as the index of the corresponding vector field. Taking this into account, our results may be compared with the results of [9] formulated in the language of vector fields. In particular, our Theorem 2 in case of quadratic vector fields explicates the general estimate of index of homogeneous vector field given in [9].

In conclusion we present a few results on discrete invariants of stable quadratic endomorphisms. As was mentioned in the introduction, for stable endomorphisms, one has an additional invariant of geometric equivalence. Namely, for each dimension n, there exists a finite list of singularity types which can occur as singularities of stable endomorphisms in dimension n [5]. The maximum of their multiplicities is called the *top stable multiplicity.* The theory of stable mappings states that, for a given stable endomorphism F, the number of stable singularities of top multiplicity $\tau_s(F)$ is finite [5]. It is also known that $\tau_s(F)$ is an invariant of geometric equivalence and one can use it to distinguish classes of stable endomorphisms. Our earlier results on algebraic formulae for topological invariants suggest that in concrete cases this invariant should be algorithmically computable [8]. We were able to prove this for dimensions $n \leq 5$.

Theorem 3. For a stable quadratic endomorphism of real vector space of dimension $n \leq 5$, the number of stable singularities of top multiplicity is equal to the signature of an explicitly constructible quadratic form.

For a stable quadratic endomorphism of the plane, singularities of top multiplicity are cusps. Using Theorem 3 one can calculate the number of cusps in concrete cases and, moreover, obtain an exact estimate for the possible spectrum of values of $\tau_s(F)$.

Proposition 2. For any stable quadratic endomorphism of the plane, the number of cusps does not exceed three and this estimate is exact.

The proof is based on case-by-case analysis of the normal forms of such endomorphisms presented in [2]. An example with three cusps is given by the formulae: $f_1 = x^2 - y^2 + x$, $f_2 = 2xy - y$.

Actually, this invariant can be used for classification of homogeneous endomorphisms as well. Let us consider all stable small perturbations of F by small linear and free terms. Let us denote by $\{\tau_s(F)\}$ the set consisting of all numbers $\tau_s(G)$ occuring in small stable deformations G of F. Obviously, this set is also an invariant of discrete equivalence. In many cases it can give more information than the mapping degree. For example, a stable small perturbation of the complex squaring mapping can have three or one cusp singularities so $\{\tau_s(F)\} = \{1, 3\}$. At the same time, for the quarto-mapping, this invariant is given by the pair $\{0, 2\}$. Hence these two endomorphisms are not geometrically equivalent. It is easy to give further examples of endomorphisms having equal mapping degrees but different invariants $\{\tau_s(F)\}$. For example, in odd dimensions the mapping degree is always zero but there exist many pairs of endomorphisms with different stabilization types. Similar problems in higher dimensions will be considered in our forthcoming publications.

მათემატიკა

კვადრატული ენდომორფიზმების დისკრეტული ინვარიანტები

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