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Mathematics

On the Estimation of the Odds-Ratio Based on Kernel Estimates of the Regression Function

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ABSTRACT. In the paper, the estimate of an odds-ratio based on the kernel estimate of the regression function is constructed. The consistency, asymptotic normality and uniform convergence of the constructed estimate are proved. *© 2017 Bull. Georg. Natl. Acad. Sci.*

Key words: odds-ratio, consistency, limit distribution, kernel estimate, uniform convergence

Let random variables $Y^{(i)}$, $i=1,2$, take two values 1 and 0 with probabilities p_i ("success") and $1-p_i$ ("failure"), $i=1,2$, respectively. Let us assume that the probability of "success" p_i is a function of the independent variable $t\in[0,1]$, that is, $p_i=p_i(t)=P\left\{Y^{(i)}=1 \,\middle|\, t\right\}$ (see. [1], [2]). Let t_i, $i=1,\dots,n$, be the points of division of the interval [0,1]: $t_i=\frac{2i-1}{2n}$, $i=1,\dots,n$. Let, further, $Y_i^{(1)}$ and $Y_i^{(2)}$, $i=1,\dots,n$, – are mutually independent Bernoulli distributed random variables with $P\left\{Y_i^{(k)}=1 \,\middle|\, t_i\right\}=p_k(t_i)$, $P\left\{Y_i^{(k)}=0 \,\middle|\, t_i\right\}=1-p_k(t_i)$, $i=1,\dots,n$, $k=1,2$. The task is to estimate the function $-(x)=\frac{p_1(x)}{(1-p_1(x))}\frac{(1-p_2(x))}{p_2(x)}$, which is called an *odds-ratio*. Such task may arise, for example, in medicine, biology ([1, 2]).

As an estimate of $¯(x)$, let consider the following statistics

$$\hat{¯}_n(x) = \frac{\hat{p}_{1n}(x)}{(1-\hat{p}_{1n}(x))} \frac{(1-\hat{p}_{2n}(x))}{\hat{p}_{2n}(x)},$$

where

$$\hat{p}_{in}(x) = p_{in}(x)\, p_n^{-1}(x), \quad i=1,2,$$

$$p_{in}(x) = \frac{1}{nb_n}\sum_{j=1}^{n} K\left(\frac{x-t_j}{b_n}\right) Y_j^{(i)},$$

$$p_n(x) = \frac{1}{nb_n}\sum_{j=1}^{n} K\left(\frac{x-t_j}{b_n}\right),$$

where $K(x)$ – certain distribution density (kernel), $\{b_n\}$ – sequence of positive numbers, converging to zero. It is clear that $0 \le \hat{p}_{in}(x) \le 1$. Certain features of estimate $\hat{p}_{in}(x)$ were studied in [3, 5].

Suppose, that the kernel $K(x)$ was chosen so that to be a function of bounded variation and satisfy the conditions $K(x)=K(-x)$, $K(x)=0$ when $|x| \ge ² > 0$, $\int K(x)dx = 1$. The class of such functions will be denoted by $H(²)$.

Lemma 1 ([5]). *Let $K(x)\in H(²)$ and $p_i(x)$, $0\le x\le 1$, $i=1,2$, is a function with the bounded variation. If $nb_n \to \infty$, then*

$$\frac{1}{nb_n}\sum_{i=1}^{n} K^{¬_1}\left(\frac{x-t_i}{b_n}\right) K^{¬_2}\left(\frac{y-t_i}{b_n}\right) p_k^{¬_3}(t_i) =$$

$$= \frac{1}{b_n}\int_0^1 K^{¬_1}\left(\frac{x-u}{b_n}\right) K^{¬_2}\left(\frac{y-u}{b_n}\right) p_k^{¬_3}(u)\,du + O\left(\frac{1}{nb_n}\right), \quad k=1,2.$$

evenly by $x, y \in [0,1]$, where $¬_i \in N \cup \{0\}$, $i=1,2,3$.

Theorem 1. *Let $K(x)$ and $p_i(x)$, $i=1,2$, satisfy the conditions of the Lemma 1. If $nb_n \to \infty$, then $\hat{¯}_n(x)$ is a consistent estimate of $¯(x)$, in points $x\in[0,1]$ of continuity $p_i(x)$ and $0 < p_i(x) < 1$, $i=1,2$.*

Proof. From Lemma 1 we have

$$Ep_{1n}(x) = \int_{(x-1)b_n^{-1}}^{x/b_n} K(t)\, p_i(x-b_n t)\,dt + O\left(\frac{1}{nb_n}\right),$$

$$p_n(x) = \frac{1}{b_n}\int_0^1 K\left(\frac{x-u}{b_n}\right) du + O\left(\frac{1}{nb_n}\right),$$

at that

$$\frac{1}{b_n}\int_0^1 K\left(\frac{x-u}{b_n}\right)du \to \text{¢}(x) = \begin{cases} 1, & x \in (0,1), \\ \frac{1}{2}, & x = 0, \quad x = 1, \end{cases}$$

$$\int_{(x-1)b_n^{-1}}^{xb_n^{-1}} K(t)\,p(x-b_n t)\,dt \to p(x)\text{¢}(x).$$

Hence, it follows that $E\hat{p}_{in}(x) \to p_i(x)$, $x \in [0,1]$ as $n \to \infty$. Analogously,

$$nb_n \operatorname{Var} \hat{p}_{in}(x) \sim \pm_i^2(x) = p_i(x)\left(1-p_i(x)\right)\int K^2(u)\,du .$$

Hence, $\hat{\text{¯}}_n(x)$ is a consistent estimate of $\text{¯}(x)$.

Theorem 2. *Let* $K(x)$ *and* $p_i(x)$, $i=1,2$, *satisfy the conditions of Lemma 1. Suppose* $x_1,\ldots,x_s$, $s \ge 1$, *are distinct points and* $0 < p_i(x_j) < 1$ *for* $j=1,\ldots,s$, $i=1,2$. *Let* $p_i'(x)$ *and* $p_i''(x)$ *exist and are bounded. If* $nb_n \to \infty$ *and* $nb_n^5 \to 0$ *as* $n \to 0$, *then* $(nb_n)^{1/2}\left(\hat{\text{¯}}_n(x_1)-\text{¯}(x_1),\ldots,\hat{\text{¯}}_n(x_s)-\text{¯}(x_s)\right)'$ *converges in distribution to* ¶, *where* ¶ *is multivariate normal variable with mean vector* 0 *and diagonal covariance matrix* $A=(a_{ij})$, *where*

$$a_{ii} = B(x_i)\int K^2(u)\,du,$$

$$B(x_i) = \text{¯}^2(x_i)\left[p_1^{-1}(x_i)\left(1-p_1(x_i)\right)^{-1} + p_2^{-1}(x_i)\left(1-p_2(x_i)\right)^{-1}\right], \quad i=1,\ldots,s.$$

Proof. For simplicity we shall prove the theorem for the case when $s=2$. Since $p_n(x) \to 1$, $x \in [0,1]$, then we will consider $p_n(x)=1$ in definition $\hat{p}_{in}(x)$.

Let's introduce the notations

$$\text{¶}_{ni}^{(k)}(x_s) = Y_i^{(k)} K\left(\frac{x_s-t_i}{b_n}\right)\frac{1}{b_n}, \quad k=1,2, \quad s=1,2, \quad i=1,\ldots,n,$$

$$\text{¦}_{ni}^{(k)}(x_s) = \text{¶}_{ni}^{(k)}(x_s) - E\text{¶}_{ni}^{(k)}(x_s),$$

$$\text{¦}_n^{(k)}(x_s) = \frac{1}{n}\sum_{i=1}^n \text{¦}_{ni}^{(k)}(x_s),$$

$$\text{¶}_n = \sqrt{nb_n}\left(\text{¦}_n^{(1)}(x_1), \text{¦}_n^{(2)}(x_1), \text{¦}_n^{(1)}(x_2), \text{¦}_n^{(2)}(x_2)\right)$$

And first of all, we will show that ¶_n converges in distribution to $\overline{\text{¶}}$, where $\overline{\text{¶}}$ fourvariate normal variable with mean 0 and diagonal covariance matrix $\overline{A} = \int K^2(u)\,du \cdot (c_{ij})$, $c_{11} = \pm_1^2(x_1)$, $c_{22} = \pm_2^2(x_1)$, $c_{33} = \pm_2^2(x_2)$, $c_{44} = \pm_2^2(x_2)$, where $\pm_i^2(x_k) = p_i(x_k)\left(1-p_i(x_k)\right)$, $i=1,2$, $k=1,2$.

To prove, we will use the theorem of Cramer-Wold ([6, Theorem xi, p. 103), it will be sufficient that $c \cdot ¶_n'$ converges in distribution to $c \cdot \bar{¶}'$ for any $c = (c_1, c_2, c_3, c_4)$ in R^4.

It follows from independence $\hat{p}_{1n}(x)$ and $\hat{p}_{2n}(x)$ that

$$\pm_n^2 = Var(c \cdot ¶_n') = nb_n \Big[c_1^2 E\left(¦_n^{(1)}(x_1)\right)^2 + c_2^2 E\left(¦_n^{(2)}(x_1)\right)^2 + c_3^2 E\left(¦_n^{(1)}(x_2)\right)^2 +$$

$$+ c_4^2 E\left(¦_n^{(2)}(x_2)\right)^2 + 2c_1c_2 E ¦_n^{(1)}(x_1) ¦_n^{(1)}(x_2) + 2c_2c_4 E ¦_n^{(2)}(x_1) ¦_n^{(2)}(x_2) \Big]. \quad (1)$$

Now, we will investigate asymptotics of items in (1). Obviously,

$$nb_n E\left(¦_n^{(1)}(x_1)\right)^2 = \frac{1}{nb_n} \sum_{i=1}^{n} p_1(t_i)\left(1 - p_1(t_i)\right) \cdot \int K^2\left(\frac{x_1 - t_i}{b_n}\right) dt .$$

Hence, owing to Lemma 1, we will receive

$$nb_n E\left(¦_n^{(1)}(x_1)\right)^2 = \frac{1}{b_n} \int_0^1 K^2\left(\frac{x_1 - t}{b_n}\right) p_1(t)\left(1 - p_1(t)\right) dt + O\left(\frac{1}{nb_n}\right)$$

Since $\left[\frac{x_1 - 1}{b_n}, \frac{x_1}{b_n}\right] \supseteq [-², ²]$, then it can be stated that

$$nb_n E\left(¦_n^{(1)}(x_1)\right)^2 = \pm_1^2(x_1) + O(b_n) + O\left(\frac{1}{nb_n}\right). \quad (2)$$

Similarly

$$nb_n E\left(¦_n^{(2)}(x_1)\right)^2 = \pm_2^2(x_1) + O(b_n) + O\left(\frac{1}{nb_n}\right),$$

$$nb_n E\left(¦_n^{(1)}(x_2)\right)^2 = \pm_1^2(x_2) + O(b_n) + O\left(\frac{1}{nb_n}\right), \quad (3)$$

$$nb_n E\left(¦_n^{(2)}(x_2)\right)^2 = \pm_2^2(x_2) + O(b_n) + O\left(\frac{1}{nb_n}\right).$$

Then, let $x_2 > x_1$, $¢ = x_2 - x_1$, $¢_n = \frac{¢}{b_n}$ and $\Pi(x) = p_1(x)\left(1 - p_1(x)\right)$. Owing to Lemma 1 we have

$$nb_n E ¦_n^{(1)}(x_1) ¦_n^{(1)}(x_2) = \int_{\frac{x_1 - 1}{b_n}}^{\frac{x_1}{b_n}} K(z) K(¢_n + z) \Pi(x_1 - b_n z) dz + O\left(\frac{1}{nb_n}\right).$$

Since $\left[\frac{x_1 - 1}{b_n}, \frac{x_1}{b_n}\right] \supseteq [-², ²]$, from this we can write

$$nb_n E\eta_n^{(1)}(x_1)\eta_n^{(1)}(x_2)=\int_{-\infty}^{\infty}K(z)K(\delta_n+z)\Pi(x_1-b_nz)dz+O\left(\frac{1}{nb_n}\right)=$$

$$=\int_{|z|<\frac{\delta_n}{2}}K(z)K(\delta_n+z)\Pi(x_1-b_nz)dz+\int_{|z|\geq\frac{\delta_n}{2}}K(z)K(\delta_n+z)\Pi(x_1-b_nz)dz+O\left(\frac{1}{nb_n}\right)\leq$$

$$\leq\sup_{|z|<\frac{\delta_n}{2}}K(\delta_n+z)\int_{-\infty}^{\infty}K(z)\Pi(x_1-b_nz)dz+\sup_{|z|>\frac{\delta_n}{2}}K(z)\int_{-\infty}^{\infty}K(\delta_n+z)\Pi(x_1-b_nz)dz+O\left(\frac{1}{nb_n}\right)\leq$$

$$\leq\sup_{|z|\geq\frac{\delta_n}{2}}K(z)\int_{-\infty}^{\infty}K(z)\Pi(x_1-b_nz)dz+\sup_{|z|\geq\frac{\delta_n}{2}}K(z)\int_{-\infty}^{\infty}K(\delta_n+z)\Pi(x_1-b_nz)dz+O\left(\frac{1}{nb_n}\right)\leq$$

$$\leq C_1\sup_{|z|\geq\frac{\delta_n}{2}}K(z)+O\left(\frac{1}{nb_n}\right)\leq C_2b_n+O\left(\frac{1}{nb_n}\right).$$

So,

$$nb_n E\eta_n^{(1)}(x_1)\eta_n^{(1)}(x_2)=O(b_n)+O\left(\frac{1}{nb_n}\right). \tag{4}$$

Similarly

$$nb_n E\eta_n^{(1)}(x_1)\eta_n^{(2)}(x_2)=O(b_n)+O\left(\frac{1}{nb_n}\right). \tag{5}$$

From (1)-(5) we can conclude that while $n\to\infty$

$$\sigma_n^2\to c\bar{A}c'>0. \tag{6}$$

Now, the only thing is to check the conditions of central limit theorem of Lyapunov. Obviously,

$$c_1\eta_n^{(1)}(x_1)+c_2\eta_n^{(2)}(x_1)+c_3\eta_n^{(1)}(x_2)+c_4\eta_n^{(2)}(x_2)=\sum_{i=1}^{n}z_{ni},$$

where

$$z_{ni}=\frac{\sqrt{b_n}}{\sqrt{n}}\left(c_1\eta_{ni}^{(1)}(x_1)+c_2\eta_{ni}^{(2)}(x_1)+c_3\eta_{ni}^{(1)}(x_2)+c_4\eta_{ni}^{(2)}(x_2)\right).$$

Let us estimate $E|z_{ni}|^{2+\delta}$, $\delta>0$. We have

$$E|z_{ni}|^{2+\delta}\leq C_3\left[\left(\frac{\sqrt{b_n}}{\sqrt{n}}\right)^{2+\delta}\left(E\left|\eta_{ni}^{(1)}(x_1)\right|^{2+\delta}+E\left|\eta_{ni}^{(2)}(x_1)\right|^{2+\delta}+E\left|\eta_{ni}^{(1)}(x_2)\right|^{2+\delta}+E\left|\eta_{ni}^{(2)}(x_2)\right|^{2+\delta}\right)\right].$$

It is easily seen that

$$\left(\frac{\sqrt{b_n}}{\sqrt{n}}\right)^{2+\delta}E\left|\eta_{ni}^{(s)}(x)\right|^{2+\delta}\leq\left(\frac{\sqrt{b_n}}{\sqrt{n}}\right)^{2+\delta}E\left|\xi_{ni}^{(s)}(x)-E\xi_{ni}^{(s)}(x)\right|^{2+\delta}\leq$$

$$\leq C_4\left(\frac{\sqrt{b_n}}{\sqrt{n}}\right)^{2+\delta}\frac{1}{b_n^{2+\delta}}K\left(\frac{x-t_i}{b_n}\right),\quad s=1,2,\quad x=x_1,\quad x=x_2.$$

Hence,

$$\sum_{i=1}^{n} E\left|z_{ni}\right|^{2+\text{¢}} \le C_5 \left(\frac{\sqrt{b_n}}{\sqrt{n}}\right)^{2+\text{¢}} \frac{nb}{b_n^{2+\text{¢}}} \left[\frac{1}{nb_n}\sum_{i=1}^{n} K\left(\frac{x_1 - t_i}{b_n}\right) + \frac{1}{nb_n}\sum_{i=1}^{n} K\left(\frac{x_2 - t_i}{b_n}\right)\right]. \qquad (7)$$

According to Lemma 1, from (7) it is easily seen that

$$R_n = \sum_{i=1}^{n} E\left|z_{ni}\right|^{2+\text{¢}} \le C_6 \frac{1}{\left(nb_n\right)^{\text{¢}/2}}.$$

From here and from (6) it appears that

$$\frac{R_n}{\pm_n} \to 0.$$

So, $\P_n$ converges in distribution to $\P$.

Let us denote

$$\bar{\xi}_n = \sqrt{nb_n}\left(\left[\frac{1}{n}\sum_{i=1}^{n}\left(\xi_{ni}^{(1)}(x_1) - p_1(x_1)\right)\right], \left[\frac{1}{n}\sum_{i=1}^{n}\left(\xi_{ni}^{(2)}(x_1) - p_2(x_1)\right)\right],\right.$$

$$\left.\left[\frac{1}{n}\sum_{i=1}^{n}\left(\P_{ni}^{(1)}(x_2) - p_1(x_2)\right)\right], \left[\frac{1}{n}\sum_{i=1}^{n}\left(\P_{ni}^{(2)}(x_2) - p_2(x_2)\right)\right]\right).$$

Let us show that $\bar{\P}_n$ converges in distribution to $\bar{\P}$.

In fact, we have

$$\P_n - \bar{\P}_n = \sqrt{nb_n}\left(\left[\frac{1}{n}\sum_{i=1}^{n}\left(E\P_{ni}^{(1)}(x_1) - p_1(x_1)\right)\right], \left[\frac{1}{n}\sum_{i=1}^{n}\left(E\P_{ni}^{(2)}(x_1) - p_2(x_1)\right)\right],\right.$$

$$\left.\left[\frac{1}{n}\sum_{i=1}^{n}\left(E\P_{ni}^{(1)}(x_2) - p_1(x_2)\right)\right], \left[\frac{1}{n}\sum_{i=1}^{n}\left(E\P_{ni}^{(2)}(x_2) - p_2(x_2)\right)\right]\right).$$

Since, according to the condition in the theorem $\int xK(x)dx = 0$, $\int x^2K(x)dx < \infty$, $p_i''(x)$, $i = 1,2$, are bounded, $nb_n \to \infty$ and $nb_n^5 \to 0$, then

$$\frac{\sqrt{b_n}}{\sqrt{n}}\left[\frac{1}{n}\sum_{i=1}^{n}\left(E\P_{ni}^{(k)}(x_s) - p_k(x_s)\right)\right] = O\left(\left(nb_n^5\right)^{1/2}\right).$$

So,

$$\P_n - \bar{\P}_n = O_p\left(\left(nb_n^5\right)^{1/2}\right).$$

We can accomplish the theorem proving, if we introduce the definition of vector function $g(y_1, y_2, y_3, y_4)$:

$$g(y_1, y_2, y_3, y_4) = \left(g_1(y_1, y_2, y_3, y_4), g_2(y_1, y_2, y_3, y_4)\right),$$

where

$$g_1(y_1, y_2, y_3, y_4) = \frac{y_1(1-y_2)}{(1-y_1)y_2},$$
$$g_2(y_1, y_2, y_3, y_4) = \frac{y_3(1-y_4)}{(1-y_3)y_4}.$$

Let $^{-} = (p_1(x_1), p_2(x_1), p_1(x_2), p_2(x_2))$. Then $\bar{¶}_n$ can be written as

$$\bar{¶}_n = (nb_n)^{1/2}(T_n - ^{-})',$$

where $T_n = (T_{n1}, T_{n2}, T_{n3}, T_{n4})$,

$$T_{n1} = \frac{1}{n}\sum_{i=1}^{n} ¶_{ni}^{(1)}(x_1), \quad T_{n2} = \frac{1}{n}\sum_{i=1}^{n} ¶_{ni}^{(2)}(x_1), \quad T_{n3} = \frac{1}{n}\sum_{i=1}^{n} ¶_{ni}^{(1)}(x_2), \quad T_{n4} = \frac{1}{n}\sum_{i=1}^{n} ¶_{ni}^{(2)}(x_2).$$

Finally, by using the theorem of Mann-Wold ([6, Theorem ii, p. 321) we can conclude that $(nb_n)^{1/2}(g(T_n) - g(^{-}))$ converges in distribution to Z, where Z is $N(D, DAD')$ and D is the matrix of partial derivatives of g, evaluated at $^{-}$. It is reality verified that $D\bar{A}D' = A$, and that

$$g(T_n) - g(^{-}) = \left(\hat{^{-}}_n(x_1) - ^{-}(x_1), \hat{^{-}}_n(x_2) - ^{-}(x_2)\right)'.$$

The theorem is proved.

Theorem 3. *Let* $K(x) \in H(^2)$ *and Fourier transform of function* $K(x)$ *is absolutely integrated, and* $p_i(x)$, $i = 1,2$, *are continuous on* $[0,1]$ *and* $0 < \inf_x p_i(x) \le \sup_x p_i(x) < 1$. *If* $nb_n^2 \to \infty$, *then* $\sup_{a \le x \le b} \left|\hat{^{-}}_n(x) - ^{-}(x)\right| \to 0$ *in probability for any* $[a,b] \subset [0,1]$.

Indeed, it is easy to be convinced that

$$\sup_{a \le x \le b} \left|\hat{^{-}}_n(x) - ^{-}(x)\right| \le 2 \sup_{a \le x \le b} \frac{|\hat{p}_{1n}(x) - p_1(x)| + |\hat{p}_{2n}(x) - p_2(x)|}{(1-\hat{p}_{1n}(x))\hat{p}_{2n}(x)(1-p_1(x))p_2(x)}.$$

Then, by using of affirmation method (x) of the item 2c.4 ([6]), we will receive

$$P\left\{\sup_{a \le x \le b} \left|\hat{^{-}}_n(x) - ^{-}(x)\right| > £\right\} \le$$
$$\le P\left\{\sup_{a \le x \le b} |\hat{p}_{1n}(x) - p_1(x)| + \sup_{a \le x \le b} |\hat{p}_{2n}(x) - p_2(x)| \ge ¢(£)\right\} + o(1), \quad ¢(£) > 0.$$

From here and from theorem 2.2 of the work [3] the proof of the theorem follows.

მათემატიკა

რეგრესიის ფუნქციის გულოვანი შეფასების საშუალებით Odds-Ratio-ს შეფასების შესახებ

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