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Algorithm for Generalized Solution of Nonlocal Boundary Value Problem

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ABSTRACT. An *m*-point nonlocal boundary value problem is posed for quasi-linear differential equations of the first order on the plane. Nonlocal boundary value problems are investigated using the algorithm of reducing nonlocal boundary value problems to a sequence of Riemann-Hilbert problems for a generalized analytic function. The conditions for the existence and uniqueness of generalized solution in the space $C_\alpha\left(\overline{G}\right)$ are considered. *© 2017 Bull. Georg. Natl. Acad. Sci.*

Key words: nonlocal boundary value problem, generalized solution

Nonlocal boundary value problems are quite an interesting generalization of classical problems and at the same time they are naturally obtained when constructing mathematical models of real processes and phenomena in physics, engineering, sociology, ecology and so on [1-8]. An *m*-point nonlocal boundary value problem of the Bitsadze-Samarski type for an elliptic equation of second order is considered in the works [9-11]. The properties of generalized analytic functions and Riemann-Hilbert boundary value problems are studied in I. Vekua's monograph [12] and in the works of G. F. Manjavidze, V. Tuchke [13]. Nonlocal boundary value problems for quasi linear differential equations of first order on the plane are considered in [14-17].

In the present paper, we pose an *m*-point nonlocal boundary value problem for quasi linear differential equations of first order on the plane. Nonlocal boundary value problems are investigated using the algorithm of reducing nonlocal boundary value problems to a sequence of Riemann-Hilbert problems for a generalized analytic function. The theorem on the existence and uniqueness of a generalized solution in the space $C_\alpha\left(\overline{G}\right)$ is proved. An *m*-point nonlocal boundary value problem for linear differential equations of first order is considered on the plane. The existence of generalized solution in the space $C_{\sim}^{p}\left(\overline{G}\right)$ is proved and a priori estimate is obtained.

1. Let G be the bounded domain on the complex plane E with the boundary Γ which is a closed simple Liapunov curve (i.e. the angle formed by the tangent to this curve with the constant direction is continuous in the Hцlder sense). We take two simple points A, B on Γ and assume that at these points there exists tangent

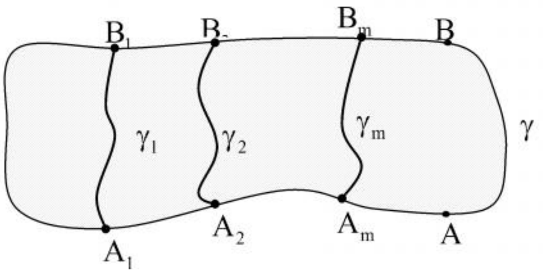

Fig. Field G.

to Γ. It is obvious that these points divide the boundary Γ into two curves. One of these parts denoted by γ is an open Liapunov curve with parametric equation $z = z(s)$, $0 \le s \le ¢$. Let us choose simple points A_k, B_k, $k = 1,\ldots,m$, on Γ\γ and assume that at these points tangent to Γ exists. Besides, we draw in G the simple smooth curves $¥_k$, $k = 1,\ldots,m$, which connect A_k and B_k. The curves $¥_k$ are assumed to have the tangents at A_k and B_k which do not coincide with the tangent to the contour Γ at the same points. It is assumed that $¥_k$ is the image of γ, diffeomorphic to $z_k = I(z)$ and with the parametric equation $z_k = z_k(s)$, $0 \le s \le ¢$, $k = 1,\ldots,m$. Furthermore, it is assumed that $¥_i \cap ¥_j = \varnothing$, $i \ne j$, $¥_i \cap ¥ = \varnothing$, $i, j = 1,\ldots,m$, and the distance between every two lines $¥_1, ¥_2, \ldots, ¥_m$ is larger than some positive number $£ = const > 0$.

Suppose that $z = x + iy \in G$, $w = w_1 + iw_2$, $\partial_{\bar z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$ is a generalized Sobolev derivative [12], $C(\bar G)$ is a Banach space consisting of all continuous functions on $\bar G$. $C_\alpha(\bar G)$ is the set of all bounded functions satisfying the Hцlder condition with index ~. The norm in $C_\alpha(\bar G)$ is defined by the equality

$$\|f\|_{C_{~}(\bar G)} = \max_{z \in \bar G}|f(z)| + \sup_{z_1, z_2 \in \bar G} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^{~}}.$$

Let us consider in $\bar G$ the following m-point nonlocal boundary value problem for quasi-linear differential equations of first order

$$\partial_{\bar z} = f(z, w, \bar w), \quad z \in G, \tag{1.1}$$

$$\begin{gathered} \operatorname{Re}[w(z)] = ¨(z), \quad z \in \Gamma \setminus ¥, \\ \operatorname{Im}\left[w\left(z^*\right)\right] = c, \quad z^* \in \Gamma \setminus ¥, \quad c = const, \end{gathered} \tag{1.2}$$

$$\operatorname{Re}\left[w(z(s))\right] = \sum_{k=1}^{m} ±_k \operatorname{Re}\left[w(z_k(s))\right], \tag{1.3}$$

$$z(s) \in ¥, \quad z_k(s) \in ¥_k$$

$$0 < ±_k = const, \quad k = 1,\ldots,m.$$

An m-point nonlocal boundary value problem of the Bitsadze-Samarski type [4] for an elliptic equation of second order is considered in the works [9-11]. Nonlocal boundary value problems for quasi-linear differential equations of first order on the plane are considered in [14-17].

For problem (1.1)-(1.3) we assume that the following conditions are fulfilled:

A1. The function $f(z, w, \bar w)$ is defined for $z \in G$, $|w| < R$, $f(z, 0, 0) \in L_p(\bar G)$, $p > 2$, and

$$|f(z, w, \bar w) - f(z, w_0, \bar w_0)| \le L(|w - w_0| + |w - \bar w_0|);$$

A2. $\text{¨}\left(z\right)\in C_{\text{~}}\left(\Gamma\setminus\text{¥}\right)$, $\text{~}>\dfrac{1}{2}$, $0<\sum_{k=1}^{m}\text{±}_k<1$.

To investigate the existence of a generalized solution of problem (1.1)-(1.3) we consider the following iteration process

$$\partial_{\overline{z}}w_n=f\left(z,w_n,\overline{w}_n\right),\quad z\in G,\tag{1.4}$$

$$\begin{gathered}\mathrm{Re}\left[w_n\left(z\right)\right]=\text{¨}\left(z\right),\quad z\in\Gamma\setminus\text{¥},\\ \mathrm{Im}\left[w_n\left(z^*\right)\right]=c,\quad z^*\in\Gamma\setminus\text{¥},\end{gathered}\tag{1.5}$$

$$\mathrm{Re}\left[w_n\left(z\left(s\right)\right)\right]=\sum_{k=1}^{m}\text{±}_k\,\mathrm{Re}\left[w_{n-1}\left(\text{z}_k\left(s\right)\right)\right],\quad z\left(s\right)\in\text{¥},\quad \text{z}_k\left(s\right)\in\text{¥}_k,\tag{1.6}$$

$$k=1,\dots,m,\quad n=1,2,3,\dots,$$

where $\mathrm{Re}\left[w_0\left(z\right)\right]$ is any function from $C_{\text{~}}\left(\text{¥}_k\right)$, $\text{~}>\dfrac{1}{2}$, $k=1,\dots,m$, that continuously adjoins the values of $\text{¨}\left(z\right)$ at the ends of the contour ¥_k.

For every $n\in N$, problem (1.4)-(1.6) is a Riemann-Hilbert type problem and its regular generalized solution belongs to the space $C_\alpha\left(\overline{G}\right)$ [12, 13].

Consider the function $v_n=w_{n+1}-w_n$. Then from (1.4)-(1.6) it follows that the function v_n is a solution of the problem

$$\partial_{\overline{z}}v_n=f\left(z,w_{n+1},\overline{w}_{n+1}\right)-f\left(z,w_n,\overline{w}_n\right)\equiv F\left(z,w_n,\overline{w}_n,w_{n+1},\overline{w}_{n+1}\right),\quad z\in G,\tag{1.7}$$

$$\begin{gathered}\mathrm{Re}\left[v_n\left(z\right)\right]=0,\quad z\in\Gamma\setminus\text{¥},\\ \mathrm{Im}\left[v_n\left(z^*\right)\right]=0,\quad z^*\in\Gamma\setminus\text{¥},\end{gathered}\tag{1.8}$$

$$\mathrm{Re}\left[v_n\left(z\left(s\right)\right)\right]=\sum_{k=1}^{m}\text{±}_k\,\mathrm{Re}\left[v_{n-1}\left(\text{z}_k\left(s\right)\right)\right],\quad z\left(s\right)\in\text{¥},\quad \text{z}_k\left(s\right)\in\text{¥}_k,\tag{1.9}$$

$$k=1,\dots,m,\quad n=1,2,3,\dots.$$

We can reduce the solution of problem (1.7)-(1.9) to the non-linear integral equation

$$v_n\left(z\right)=\text{·}_n\left(z\right)+\text{¤}_n\left(z\right)-\frac{1}{\text{®}}\iint_G\frac{F\left(\text{s},w_n\left(\text{¸}\right),\overline{w}_n\left(\text{¸}\right),w_{n+1}\left(\text{¸}\right),\overline{w}_{n+1}\left(\text{¸}\right)\right)}{\text{¸}-z}\,d\text{¶}\,d\text{¦},\tag{1.10}$$

where $\text{¸}=\text{¶}+i\text{¦}$, $\text{·}_n\left(z\right)$ is a holomorphic function satisfying conditions (1.8)-(1.9) and $\text{¤}_n\left(z\right)$ is a holomorphic function such that the difference

$$\text{¤}_n\left(z\right)-\frac{1}{\text{®}}\iint_G\frac{F\left(\text{s},w_n\left(\text{¸}\right),\overline{w}_n\left(\text{¸}\right),w_{n+1}\left(\text{¸}\right),\overline{w}_{n+1}\left(\text{¸}\right)\right)}{\text{¸}-z}\,d\text{¶}\,d\text{¦}$$

satisfies homogeneous boundary conditions.

The integral operator in the right-hand part of equation (1.10) is denoted by

$$T_G\left[z,F\right]=-\frac{1}{\text{®}}\iint_G \frac{F\left(\text{s},w_n\left(\text{¸}\right),\overline{w}_n\left(\text{¸}\right),w_{n+1}\left(\text{¸}\right),\overline{w}_{n+1}\left(\text{¸}\right)\right)}{\text{¸}-z}\,d\text{¶}\,d\text{¦},\quad \text{¸}=\text{¶}+i\text{¦}\ .$$

The operator T_G maps the space $L_p\left(\overline{G}\right)$ into $C\ \left(\overline{G}\right)$, $\ =\frac{p-2}{p}<\text{~}$.

The following theorem holds true:

Theorem 1. *Let the following conditions be fulfilled:*

(i) *the function* $f\left(z,w,\overline{w}\right)$ *is defined for* $z\in G$, $|w|<R$, $f\left(z,0,0\right)\in L_p\left(\overline{G}\right)$, $p>2$, *and*

$$\left|f\left(z,w,\overline{w}\right)-f\left(z,w_0,\overline{w}_0\right)\right|\le L\left(\left|w-w_0\right|+\left|w-\overline{w}_0\right|\right);$$

(ii) $\text{¨}\left(z\right)\in C_{\text{~}}\left(\Gamma\setminus\text{¥}\right)$ $\ \text{~}>\frac{1}{2},\ 0<\sum_{k=1}^{m}\text{±}_k<1,\ 0<\text{±}_k=const,\ k=1,\ldots,m;$

(iii) *there exists a number* $R_1>0$, $R_1\le R$, *such that the inequality*

$$\left\|\cdot\ _n\right\|_{C_{\text{~}}\left(\overline{G}\right)}+\left(C_1+\left\|T_G\right\|_{L_p\left(\overline{G}\right),C_{\text{~}}\left(\overline{G}\right)}\right)\left(2L\left|G\right|^{1/p}R_1\right)\le R_1,$$

where $\left|G\right|=\text{mes}\,G$,

$$2\left|G\right|^{1/p}L\left(C_1+\left\|T_G\right\|_{L_p\left(\overline{G}\right),C_{\text{~}}\left(\overline{G}\right)}\right)<1$$

is fulfilled. Then the solution of problem (1.1)-(1.3) exists in the space $C_{\text{~}}\left(\overline{G}\right)$ and is unique.

2. Consider in the domain $\overline{G}$ the following *m*-point nonlocal boundary value problem for a linear differential equation of first order

$$\begin{gathered}\partial_{\overline{z}}w=A\left(z\right)w+A\left(z\right)\overline{w}+d\left(z\right),\quad z\in G,\\ \text{Re}\left[w\left(z\right)\right]=0,\quad z\in\Gamma\setminus\text{¥},\\ \text{Im}\left[w\left(z^*\right)\right]=0,\quad z^*\in\Gamma\setminus\text{¥},\\ \text{Re}\left[w\left(z\left(s\right)\right)\right]=\sum_{k=1}^{m}\text{±}_k\,\text{Re}\left[w\left(z_k\left(s\right)\right)\right],\quad z\left(s\right)\in\text{¥},\quad z\left(s\right)\in\text{¥}_k,\\ 0<\text{±}_k=const,\quad k=1,\ldots,m.\end{gathered}\tag{2.1}$$

Assume that $A\left(z\right),B\left(z\right),d\left(z\right)\in L_p\left(\overline{G}\right),\ p>2,\ |A|,|B|\le N.$

Denote by $C_{\text{~}}^p\left(\overline{G}\right)$ the set of functions $w\left(z\right)\in L_{\text{~}}\left(\overline{G}\right)$ such that

$$\begin{gathered}\text{Re}\left[w\left(z\right)\right]=0,\quad z\in\Gamma\setminus\text{¥},\\ \text{Im}\left[w\left(z^*\right)\right]=0,\quad z^*\in\Gamma\setminus\text{¥},\\ \text{Re}\left[w\left(z\left(s\right)\right)\right]=\sum_{k=1}^{m}\text{±}_k\,\text{Re}\left[w\left(z_k\left(s\right)\right)\right],\quad z\left(s\right)\in\text{¥},\quad z\left(s\right)\in\text{¥}_k,\quad k=1,\ldots,m.\end{gathered}\tag{2.2}$$

and having the finite norm

$$\|w\|_{C_{\sim}^{p}(\bar{G})} = \|w\|_{C_{\sim}(\bar{G})} + \|\partial_{\bar{z}} w\|_{L_p(\bar{G})} < +\infty . \tag{2.3}$$

The set $C_{\sim}^{p}(\bar{G})$ is a linear normed space over the real field with the norm defined by means of equality (2.3). If $p > q > 2$, then $C_{\sim}^{p}(\bar{G}) \supset C_{\sim}^{q}(\bar{G})$ and $\|w\|_{C_{\sim}^{q}(\bar{G})} \le \ell \|w\|_{C_{\sim}^{p}(\bar{G})}$, where ℓ is a positive constant and w is any element from $C_{\sim}^{p}(\bar{G})$. The following theorem holds true.

Theorem 2. *For any function* $d(z) \in L_p(\bar{G})$, $p > 2$, *a solution* $w(z)$ *of problem* (2.1) *exists, belongs to the space* $C_{\sim}^{p}(\bar{G})$ *and the following a priori estimate holds for it*

$$\|w\|_{C_{\sim}^{p}(\bar{G})} \le \mathrm{a} \|d\|_{L_p(\bar{G})} ,$$

where a is the positive constant depending only on p, N and $|G| = \text{mes}\, G$.

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ნაშრომში განხილულია m-წერტილოვანი არალოკალური სასაზღვრო ამოცანა პირველი რიგის კვაზიწრფივი დიფერენციალური განტოლებებისათვის სიბრტყეზე და ამოხსნის იტერაციული ალგორითმი. დამტკიცებულია თეორემები განზოგადოებული ამონახსნის არსებობისა და ერთადერთობის შესახებ.

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