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Mathematics

On Investigation of Dynamical Three-Dimensional Model of Thermoelastic Piezoelectric Solids

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ABSTRACT. In the present paper dynamical three-dimensional model of thermoelastic piezoelectric solid consisting of anisotropic inhomogeneous material with regard to magnetic field is considered. Initial-boundary value problem corresponding to the dynamical model is investigated, where on certain parts of the boundary displacement, electric and magnetic potentials, and temperature vanish, and on the remaining parts components of stress vector, electric displacement and magnetic induction, and heat flux along the outward normal vector of the boundary are given. The variational formulation of the initial-boundary value problem is obtained, which is equivalent to the differential formulation of the three-dimensional initial-boundary value problem in the spaces of smooth enough functions. Applying the variational formulation existence, uniqueness and continuous dependence of solution on the given data is proved in suitable function spaces. *© 2017 Bull. Georg. Natl. Acad. Sci.*

Key words: piezoelectric thermoelastic solids, initial-boundary value problem, variational formulation, existence and uniqueness of solution

Piezoelectric materials are widely used and intensively being investigated for possible application as adaptive materials, which enable to change their shape or material characteristics, and thereby they can replace mechanical actuators and sensors in modern engineering structures. One of the theoretical models of piezoelectricity was developed by W. Voigt [1], which describes the interaction between elastic, electric and thermal properties of an elastic body. Subsequently, W. Cady [2] treated the physical properties of piezoelectric crystals as well as their practical applications. H. Tiersten [3] studied problems of vibration of piezoelectric plates. The widespread use of adaptive materials in diverse engineering construction, in particular, in aerospace industry, where sensors and actuators might undergo high thermal as well as mechanical stresses, has activated researches on thermal along with the mechanical and electro-magnetic properties of materials. A three-dimensional model of thermoelastic piezoelectric bodies was derived by R. Mindlin [4] on the basis of variational principle. Further, W. Nowacki [5] developed some general theorems for thermoelastic piezoelectric materials. R. Dhaliwal and J. Wang [6] proved uniqueness theorem for linear three-dimensional model of the theory of thermo-piezoelectricity, which was generalized by J. Li in the paper [7], where a generalization of the reciprocity theorem of Nowacki [8] was also obtained. Applying the potential method and the theory of integral equations D. Natroshvili [9] studied problems of statics and pseudo-oscillations with basic and crack type boundary conditions for thermoelastic piezolelectric bodies with regard to magnetic field consisting of homogeneous material.

It should be pointed out that three-dimensional initial-boundary value problems with general mixed boundary conditions for displacement, electric and magnetic potentials, and temperature corresponding to the linear dynamical models for inhomogeneous anisotropic thermoelastic piezoelectric bodies with regard to magnetic field have not been investigated yet. The well-posedness results are mainly obtained for thermoelastic piezoelectric bodies consisting of homogeneous materials. In this paper we study three-dimensional dynamical model for thermoelastic piezoelectric solid consisting of anisotropic inhomogeneous material with regard to magnetic field. We consider initial-boundary value problem with general mixed boundary conditions corresponding to the three-dimensional model, where on certain parts of the boundary displacement, electric and magnetic potentials, and temperature vanish, and on the remaining parts components of stress vector, electric displacement and magnetic induction, and heat flux along the outward normal vector of the boundary are given. We obtain variational formulation of the initial-boundary value problem, which is equivalent to the differential formulation in the spaces of smooth enough functions. On the basis of the variational formulation we obtain the existence, uniqueness and continuous dependence results in suitable spaces of vector-valued distributions with values in factor spaces of Sobolev spaces.

We denote by $W^{r,2}(D)=H^r(D)$ and $H^r(\hat{\Gamma})$, $r\geq 1$, $r\in\mathbf{R}$, the Sobolev spaces of order r based on the spaces $H^0(D)=L^2(D)$ and $H^0(D)=L^2(D)$ of square-integrable functions, respectively, where $D\subset\mathbf{R}^p$, $p\in\mathbf{N}$, is a bounded Lipschitz domain [10] and $\hat{\Gamma}\subset\partial D$ is a Lipschitz surface. We denote by $\mathbf{H}^r(D)=[H^r(D)]^3$, $\mathbf{L}^2(D)=[L^2(D)]^3$, $\mathbf{L}^s(\hat{\Gamma})=[L^s(\hat{\Gamma})]^3$, $s\geq 1$, $s\in\mathbf{R}$, the corresponding spaces of vector-valued functions. The trace operators we denote by $tr_{\hat{\Gamma}}:H^1(D)\to H^{1/2}(\hat{\Gamma})$ and $\mathbf{tr}_{\hat{\Gamma}}:\mathbf{H}^1(D)\to\mathbf{H}^{1/2}(\hat{\Gamma})$. We denote by $C^{q,1}(D)$ the space of functions on D with Lipschitz continuous derivatives up to the order $q\in\mathbf{N}\cup\{0\}$. For Banach space X, $C([0,T];X)$ denotes the space of continuous functions on $[0,T]$ with values in X, $L^q(0,T;X)$, $1\leq q\leq\infty$, is the space of such functions $g:(0,T)\to X$ that $\|g(t)\|_X\in L^q(0,T)$. We denote by $g'=dg/dt$ and $g''=d^2g/dt^2$ the generalized first and second order derivatives of function $g\in L^q(0,T;X)$ [11].

Let us consider a thermoelastic piezoelectric body with initial configuration $\overline{\Omega}\subset\mathbf{R}^3$, which consists of general inhomogeneous anisotropic material. The body is clamped along a part $\Gamma_0\subset\Gamma=\partial\Omega$ of the Lipschitz boundary $\partial\Omega$ and on the remaining part $\Gamma_1=\Gamma\setminus\overline{\Gamma_0}$ surface force with density $\mathbf{g}=(g_i):\Gamma_1\times(0,T)\to\mathbf{R}^3$ is given, $\partial\Omega=\Gamma_0\cup\Gamma_{01}\cup\Gamma_1$, $\Gamma_0\cap\Gamma_1=\varnothing$, is a Lipschitz dissection [10] of $\partial\Omega$; electric potential ϕ vanishes along $\Gamma_0^\phi\subset\Gamma$ and on the remaining part $\Gamma_1^\phi=\Gamma\setminus\overline{\Gamma_0^\phi}$ of the boundary the normal component of the electric displacement with density $g^\phi:\Gamma_1^\phi\times(0,T)\to\mathbf{R}$ is given, where $\partial\Omega=\Gamma_0^\phi\cup\Gamma_{01}^\phi\cup\Gamma_1^\phi$, $\Gamma_0^\phi\cap\Gamma_1^\phi=\varnothing$, is a Lipschitz dissection of $\partial\Omega$; magnetic potential ψ vanishes along $\Gamma_0^\psi\subset\Gamma$ and on the remaining part

$\Gamma_1^\psi = \Gamma \setminus \overline{\Gamma_0^\psi}$ of the boundary the normal component of the magnetic induction with density $g^\psi : \Gamma_1^\psi \times (0,T) \to \mathbf{R}$ is given, where $\partial\Omega = \Gamma_0^\psi \cup \Gamma_{01}^\psi \cup \Gamma_1^\psi$, $\Gamma_0^\psi \cap \Gamma_1^\psi = \varnothing$, is a Lipschitz dissection of $\partial\Omega$; temperature θ vanishes along $\Gamma_0^\theta \subset \Gamma$ and on the remaining part $\Gamma_1^\theta = \Gamma \setminus \overline{\Gamma_0^\theta}$ of the boundary the normal component of heat flux with density $g^\theta : \Gamma_1^\theta \times (0,T) \to \mathbf{R}$ is given. The dynamical three-dimensional model of the thermoelastic piezoelectric body Ω in differential form with quasi-static equations for electro-magnetic fields, where the rate of magnetic field is small, i.e. electric field is curl free, and there is no electric current, i.e. magnetic field is curl free, is given by following initial-boundary value problem [7, 9]:

$$\rho\frac{\partial^2 u_i}{\partial t^2} - \sum_{j=1}^{3}\frac{\partial \sigma_{ij}}{\partial x_j} = f_i \text{ in } \Omega\times(0,T),\ i = 1,2,3, \tag{1}$$

$$\sum_{j=1}^{3}\frac{\partial D_j}{\partial x_j} = f^\varepsilon \text{ in } \Omega\times(0,T), \tag{2}$$

$$\sum_{j=1}^{3}\frac{\partial B_j}{\partial x_j} = 0 \text{ in } \Omega\times(0,T), \tag{3}$$

$$\chi\frac{\partial\theta}{\partial t} - \sum_{j=1}^{3}\frac{\partial}{\partial x_i}\left(\eta_{ij}\frac{\partial\theta}{\partial x_j}\right) + \Theta_0\frac{\partial}{\partial t}\sum_{i,j=1}^{3}\lambda_{ij}e_{ij}(\mathbf{u})$$

$$-\Theta_0\frac{\partial}{\partial t}\sum_{i=1}^{3}\mu_i\frac{\partial\phi}{\partial x_i} - \Theta_0\frac{\partial}{\partial t}\sum_{i=1}^{3}m_i\frac{\partial\psi}{\partial x_i} = f^\theta \text{ in } \Omega\times(0,T), \tag{4}$$

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma_0\times(0,T), \qquad \sum_{j=1}^{3}\sigma_{ij}n_j = g_i \text{ on } \Gamma_1\times(0,T),\ i = 1,2,3, \tag{5}$$

$$\phi = 0 \text{ on } \Gamma_0^\phi\times(0,T), \qquad \sum_{i=1}^{3}D_i n_i = g^\phi \text{ on } \Gamma_1^\phi\times(0,T), \tag{6}$$

$$\psi = 0 \text{ on } \Gamma_0^\psi\times(0,T), \qquad \sum_{i=1}^{3}B_i n_i = g^\psi \text{ on } \Gamma_1^\psi\times(0,T), \tag{7}$$

$$\theta = 0 \text{ on } \Gamma_0^\theta\times(0,T), \qquad -\sum_{i,j=1}^{3}\eta_{ij}\frac{\partial\theta}{\partial x_j}n_i = g^\theta \text{ on } \Gamma_1^\theta\times(0,T), \tag{8}$$

$$u_i(x,0) = u_{0i}(x),\ \frac{\partial u_i}{\partial t}(x,0) = u_{1i}(x),\ \theta(x,0) = \theta_0(x) \text{ in } \Omega,\ i = 1,2,3, \tag{9}$$

where $\mathbf{n} = (n_i)_{i=1}^3$ is the unit outward normal vector to Γ, $\mathbf{u} = (u_i) : \Omega\times(0,T) \to \mathbf{R}^3$ is the displacement vector-function, ρ is the mass density in the reference configuration, $\phi : \Omega\times(0,T) \to \mathbf{R}$ and $\psi : \Omega\times(0,T) \to \mathbf{R}$ stand for the electric and magnetic potentials such that the electric and magnetic fields are $\mathbf{E} = -grad\phi$ and $\mathbf{H} = -grad\psi$, $\theta : \Omega\times(0,T) \to \mathbf{R}$ is the temperature distribution,

$\mathbf{f}=(f_i)_{i=1}^3:\Omega\times(0,T)\to\mathbf{R}^3$ is the density of applied body forces, $f^\varepsilon:\Omega\times(0,T)\to\mathbf{R}$ is the density of electric charges, and $f^\theta:\Omega\times(0,T)\to\mathbf{R}$ is the density of heat sources, $\mathbf{u}_0=(u_{0i})$ and $\mathbf{u}_1=(u_{1i})$ are the initial displacement and velocity vector-functions, θ_0 is the initial distribution of temperature. $(\sigma_{ij})_{i,j=1}^3$ is the mechanical stress tensor, $\mathbf{D}=(D_j)_{j=1}^3$ is the electric displacement vector, and $\mathbf{B}=(B_j)_{j=1}^3$ is the magnetic induction vector, which are given by the following constitutive equations:

$$\sigma_{ij}=\sum_{p,q=1}^3 c_{ijpq}e_{pq}(\mathbf{u})+\sum_{p=1}^3\varepsilon_{pij}\frac{\partial\phi}{\partial x_p}+\sum_{p=1}^3 b_{pij}\frac{\partial\psi}{\partial x_p}-\lambda_{ij}\theta,\qquad i,j=1,2,3,$$

$$D_i=\sum_{p,q=1}^3\varepsilon_{ipq}e_{pq}(\mathbf{u})-\sum_{j=1}^3 d_{ij}\frac{\partial\phi}{\partial x_j}-\sum_{j=1}^3 a_{ij}\frac{\partial\psi}{\partial x_j}+\mu_i\theta,\ i=1,2,3,$$

$$B_i=\sum_{p,q=1}^3 b_{ipq}e_{pq}(\mathbf{u})-\sum_{j=1}^3 a_{ij}\frac{\partial\phi}{\partial x_j}-\sum_{j=1}^3\zeta_{ij}\frac{\partial\psi}{\partial x_j}+m_i\theta,\ i=1,2,3,$$

where $e_{ij}(\mathbf{v})=1/2\left(\partial v_i/\partial x_j+\partial v_j/\partial x_i\right)$, $i,j=1,2,3$, $\mathbf{v}=(v_i)_{i=1}^3$, is the strain tensor, $(c_{ijpq})_{i,j,p,q=1}^3$ is the elasticity tensor, $(\varepsilon_{pij})_{i,j,p=1}^3$ are piezoelectric and $(b_{pij})_{i,j,p=1}^3$ are piezomagnetic coefficients, $(\lambda_{ij})_{i,j=1}^3$ is the stress-temperature tensor, $(d_{ij})_{i,j=1}^3$ and $(\zeta_{ij})_{i,j=1}^3$ are the permittivity and permeability tensors, $(a_{ij})_{i,j=1}^3$ are the coupling coefficients connecting electric and magnetic fields, $(\mu_i)_{i=1}^3$ and $(m_i)_{i=1}^3$ are coefficients characterizing the relation between thermal, electric and magnetic fields, $\Theta_0>0$ is the temperature of thermoelastic body in natural state in the absence of deformation and electromagnetic fields, which is considered as a reference temperature, $(\eta_{ij})_{i,j=1}^3$ is the thermal conductivity tensor and χ is the thermal capacity. We assume that the elasticity tensor, piezoelectric and piezomagnetic coefficients, and the stress-temperature tensor satisfy the following symmetry conditions

$$c_{ijpq}=c_{ijqp}=c_{jipq},\ \varepsilon_{pij}=\varepsilon_{pji},\ b_{pij}=b_{pji},\ d_{ij}=d_{ji},\ a_{ij}=a_{ji},\ \zeta_{ij}=\zeta_{ji},\ \lambda_{ij}=\lambda_{ji},\ i,j,p,q=1,2,3. \quad (10)$$

To investigate the existence and uniqueness of weak solution of the three-dimensional initial-boundary value problem (1)-(9) we consider the following variational formulation, which is equivalent to the differential formulation in spaces of smooth enough functions: Find the unknown vector-function $\mathbf{u}\in C([0,T];\mathbf{V}(\Omega))$, $\mathbf{u}'\in L^\infty(0,T;\mathbf{V}(\Omega))$, $\mathbf{u}''\in L^\infty(0,T;\mathbf{L}^2(\Omega))$, and functions $\phi\in C([0,T];V^\phi(\Omega))$, $\phi'\in L^\infty(0,T;V^\phi(\Omega))$, $\psi\in C([0,T];V^\psi(\Omega))$, $\psi'\in L^\infty(0,T;V^\psi(\Omega))$, $\theta\in C([0,T];V^\theta(\Omega))$, $\theta'\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;V^\theta(\Omega))$, which satisfy the following equations in the sense of distributions on $(0,T)$,

$$(\rho\mathbf{u}'',\mathbf{v})_{\mathbf{L}^2(\Omega)}+c(\mathbf{u},\mathbf{v})+\varepsilon(\phi,\mathbf{v})+b(\psi,\mathbf{v})-\lambda(\theta,\mathbf{v})=L^{\mathbf{u}}(\mathbf{v}),\qquad\forall\mathbf{v}\in\mathbf{V}(\Omega), \quad (11)$$

$$-\varepsilon(\bar\phi,\mathbf{u})+d(\phi,\bar\phi)+a(\psi,\bar\phi)-\mu(\theta,\bar\phi)=L^\phi(\bar\phi),\qquad\forall\bar\phi\in V^\phi(\Omega), \quad (12)$$

$$-b(\bar\psi,\mathbf{u})+a(\phi,\bar\psi)+\zeta(\psi,\bar\psi)-m(\theta,\bar\psi)=L^\psi(\bar\psi),\qquad\forall\bar\psi\in V^\psi(\Omega), \quad (13)$$

$$(\chi\theta',\bar\theta)_{L^2(\Omega)}+\eta(\theta,\bar\theta)+\Theta_0\lambda(\bar\theta,\mathbf{u}')-\Theta_0\mu(\bar\theta,\phi')-\Theta_0 m(\bar\theta,\psi')=L^\theta(\bar\theta),\quad\forall\bar\theta\in V^\theta(\Omega), \quad (14)$$

together with the initial conditions

$$\mathbf{u}(0)=\mathbf{u}_0,\ \ \mathbf{u}'(0)=\mathbf{u}_1,\ \ \theta(0)=\theta_0, \tag{15}$$

where $\mathbf{V}(\Omega)=\{\mathbf{v}\in\mathbf{H}^1(\Omega);\ \mathbf{tr}_\Gamma(\mathbf{v})=\mathbf{0}$ on $\Gamma_0\}$, $V^\phi(\Omega)=\{\overline{\phi}\in H^1(\Omega);\ tr_\Gamma(\overline{\phi})=0$ on $\Gamma_0^\phi\}$, $V^\psi(\Omega)=\{\overline{\psi}\in H^1(\Omega);\ tr_\Gamma(\overline{\psi})=0$ on $\Gamma_0^\psi\}$, $V^\theta(\Omega)=\{\overline{\theta}\in H^1(\Omega);\ tr_\Gamma(\overline{\theta})=0$ on $\Gamma_0^\theta\}$,

$$c(\mathbf{u},\mathbf{v})=\int\limits_\Omega\sum_{i,j,p,q=1}^{3}c_{ijpq}e_{pq}(\mathbf{u})e_{ij}(\mathbf{v})dx,\quad \varepsilon(\phi,\mathbf{v})=\int\limits_\Omega\sum_{i,j,p=1}^{3}\varepsilon_{pij}\frac{\partial\phi}{\partial x_p}e_{ij}(\mathbf{v})dx,$$

$$b(\psi,\mathbf{v})=\int\limits_\Omega\sum_{i,j,p=1}^{3}b_{pij}\frac{\partial\psi}{\partial x_p}e_{ij}(\mathbf{v})dx,\quad \lambda(\theta,\mathbf{v})=\int\limits_\Omega\sum_{i,j=1}^{3}\lambda_{ij}\theta e_{ij}(\mathbf{v})dx,$$

$$d(\phi,\overline{\phi})=\int\limits_\Omega\sum_{i,j=1}^{3}d_{ij}\frac{\partial\phi}{\partial x_j}\frac{\partial\overline{\phi}}{\partial x_i}dx,\quad a(\psi,\overline{\phi})=\int\limits_\Omega\sum_{i,j=1}^{3}a_{ij}\frac{\partial\psi}{\partial x_j}\frac{\partial\overline{\phi}}{\partial x_i}dx,$$

$$\mu(\theta,\overline{\phi})=\int\limits_\Omega\sum_{i=1}^{3}\mu_i\theta\frac{\partial\overline{\phi}}{\partial x_i}dx,\quad \zeta(\psi,\overline{\psi})=\int\limits_\Omega\sum_{i,j=1}^{3}\zeta_{ij}\frac{\partial\psi}{\partial x_j}\frac{\partial\overline{\psi}}{\partial x_i}dx,$$

$$m(\theta,\overline{\psi})=\int\limits_\Omega\sum_{i=1}^{3}m_i\theta\frac{\partial\overline{\psi}}{\partial x_i}dx,\quad \eta(\theta,\overline{\theta})=\int\limits_\Omega\sum_{i,j=1}^{3}\eta_{ij}\frac{\partial\theta}{\partial x_j}\frac{\partial\overline{\theta}}{\partial x_i}dx,$$

$$L^{\mathbf{u}}(\mathbf{v})=\int\limits_\Omega\sum_{i=1}^{3}f_iv_idx+\int\limits_{\Gamma_1}\sum_{i=1}^{3}g_itr_{\Gamma_1}(v_i)d\Gamma,\quad L^\phi(\overline{\phi})=\int\limits_\Omega f^\varepsilon\overline{\phi}dx-\int\limits_{\Gamma_1^\phi}g^\phi tr_{\Gamma_1^\phi}(\overline{\phi})d\Gamma,$$

$$L^\psi(\overline{\psi})=-\int\limits_{\Gamma_1^\psi}g^\psi tr_{\Gamma_1^\psi}(\overline{\psi})d\Gamma,\quad L^\theta(\overline{\theta})=\int\limits_\Omega f^\theta\overline{\theta}dx-\int\limits_{\Gamma_1^\theta}g^\theta tr_{\Gamma_1^\theta}(\overline{\theta})d\Gamma,$$

$(.,.)_{\mathbf{L}^2(\Omega)}$ and $(.,.)_{L^2(\Omega)}$ are scalar products in the spaces $\mathbf{L}^2(\Omega)$ and $L^2(\Omega)$, respectively.

Note that if the parts Γ_0^ϕ and Γ_0^ψ of the body Ω, where electric and magnetic potentials vanish, are empty sets, then the homogeneous problem (11)-(15) has non-trivial solutions. Indeed, if the tensors $(d_{ij})_{i,j=1}^3$, $(a_{ij})_{i,j=1}^3$ and $(\zeta_{ij})_{i,j=1}^3$ characterizing electric and magnetic fields satisfy the following inequality

$$\sum_{i,j=1}^{3}d_{ij}\xi_j\xi_i+2\sum_{i,j=1}^{3}a_{ij}\overline{\xi}_j\xi_i+\sum_{i,j=1}^{3}\zeta_{ij}\overline{\xi}_j\overline{\xi}_i\geq\beta\sum_{i=1}^{3}\left((\xi_i)^2+(\overline{\xi}_i)^2\right),\qquad \forall\xi_i,\overline{\xi}_i\in\mathbf{R},\ \beta=const>0,$$

and $\mathbf{u}=\mathbf{0}$, $\theta=0$, $f^\varepsilon=0$, $g^\phi=0$, $g^\psi=0$, then the solutions ϕ and ψ are constants. Hence, the solution of the problem (11)-(15) is not unique in the spaces mentioned in the variational formulation and it is necessary to introduce suitable factor spaces, where the solution of the problem (11)-(15) will be unique. Let us denote by $R_\phi=\{v\in V^\phi(\Omega);v=\alpha_\phi,\alpha_\phi=const\}$ and $R_\psi=\{v\in V^\psi(\Omega);v=\alpha_\psi,\alpha_\psi=const\}$ the subspaces of $V^\phi(\Omega)$ and $V^\psi(\Omega)$, which correspond to the homogeneous problem. Applying them we introduce the factor spaces $V^\phi(\Omega)/R_\phi$ and $V^\psi(\Omega)/R_\psi$, which consist of equivalence classes $\overline{\phi}^{R_\phi}=\{\overline{\phi}+\phi^r;\phi^r\in R_\phi\}$, for each $\overline{\phi} \in V^{\phi}(\Omega)$, and $\overline{\psi}^{R_\psi} = \{\overline{\psi} + \psi^r ; \psi^r \in R_\psi\}$, for each $\overline{\psi} \in V^{\psi}(\Omega)$, respectively. The spaces $V^{\phi}(\Omega)/R_\phi$ and $V^{\psi}(\Omega)/R_\psi$ are the Hilbert spaces with respect to the norms $\| \overline{\phi}^{R_\phi} \|_{V^\phi(\Omega)/R_\phi} = \inf\{\| \overline{\phi} + \phi^r \|_{H^1(\Omega)} ; \phi^r \in R_\phi\}$ and $\| \overline{\psi}^{R_\psi} \|_{V^\psi(\Omega)/R_\psi} = \inf\{\| \overline{\psi} + \psi^r \|_{H^1(\Omega)} ; \psi^r \in R_\psi\}$.

If $(\mathbf{u},\phi,\psi,\theta)$ is a solution of the problem (11)-(15), then any function $(\mathbf{u},\phi,\psi,\theta)+(0,\phi^r,\psi^r,0)$, where $\phi^r \in R_\phi$, $\psi^r \in R_\psi$, is a solution of (11)-(15). Therefore, we say that $(\mathbf{u},\phi^{R_\phi},\psi^{R_\psi},\theta)$ is a solution of the problem (11)-(15), if any function from the equivalence class $(\mathbf{u},\phi^{R_\phi},\psi^{R_\psi},\theta)$ is a solution of the problem (11)-(15).

For the initial-boundary value problem (11)-(15) corresponding to the dynamical three-dimensional model for thermoelastic piezoelectric body with regard to magnetic field the following theorem is valid.

Theorem. *Suppose that* $\Omega \subset \mathbf{R}^3$ *is a bounded domain with Lipschitz boundary and the parameters characterizing thermo-mechanical and electro-magnetic properties of the body* Ω *are such that* $\rho, \chi \in L^\infty(\Omega)$, $\rho(x) > \alpha_\rho = const > 0$, $\chi(x) > \alpha_\chi = const > 0$, *for almost all* $x \in \Omega$, c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, $m_i \in L^\infty(\Omega)$, $\eta_{ij} \in C^{0,1}(\Omega)$, $i,j,p,q = 1,2,3$ *and satisfy symmetry conditions* (10) *and the following positive definiteness conditions*

$$\sum_{i,j,p,q=1}^{3} c_{ijpq}\xi_{ij}\xi_{pq} \geq \alpha_c \sum_{i,j=1}^{3} (\xi_{ij})^2, \;\; \forall \xi_{ij} \in \mathbf{R}, \xi_{ij} = \xi_{ji}, \qquad \sum_{i,j=1}^{3} \eta_{ij}\xi_j\xi_j \geq \alpha_\eta \sum_{i=1}^{3} (\xi_i)^2, \;\; \forall \xi_i \in \mathbf{R},$$

$$\sum_{i,j=1}^{3} d_{ij}\xi_j\xi_i + 2\sum_{i,j=1}^{3} a_{ij}\overline{\xi}_j\xi_i + \sum_{i,j=1}^{3} \zeta_{ij}\overline{\xi}_j\overline{\xi}_i + \frac{1}{\Theta_0}\chi\xi\xi$$
$$-2\sum_{i=1}^{3} \mu_i\xi\xi_i - 2\sum_{i=1}^{3} m_i\xi\overline{\xi}_i \geq \alpha \sum_{i=1}^{3} \left((\xi_i)^2 + (\overline{\xi}_i)^2 + \xi^2 \right), \qquad \forall \xi_i, \overline{\xi}_i, \xi \in \mathbf{R},$$

for almost all $x \in \Omega$, *where* α_c, α_η, α *are positive constants. If* $\mathbf{u}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{V}(\Omega)$, $\mathbf{u}_1 \in \mathbf{V}(\Omega)$, $\theta_0 \in H^2(\Omega) \cap V^\theta(\Omega)$, $\mathbf{f}, \mathbf{f}' \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g}, \mathbf{g}', \mathbf{g}'' \in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, $f^\varepsilon, (f^\varepsilon)', (f^\varepsilon)'' \in L^2(0,T;L^{6/5}(\Omega))$, $g^\phi, (g^\phi)', (g^\phi)'' \in L^2(0,T;L^{4/3}(\Gamma_1^\phi))$, $g^\psi, (g^\psi)', (g^\psi)'' \in L^2(0,T;L^{4/3}(\Gamma_1^\psi))$, $f^\theta, (f^\theta)' \in L^2(0,T;L^{6/5}(\Omega))$, $g^\theta, (g^\theta)' \in L^2(0,T;L^{4/3}(\Gamma_1^\theta))$, *and* $L^\phi(\phi^r) = 0$, $L^\psi(\psi^r) = 0$, $\forall \phi^r \in R_\phi, \psi^r \in R_\psi$,

$$g^\theta(0) = -\sum_{i,j=1}^{3} \eta_{ij} tr_{\Gamma_1^\theta}\left(\frac{\partial \theta_0}{\partial x_j}\right) n_i \quad \text{on } \Gamma_1^\theta,$$

where $\mathbf{n} = (n_i)_{i=1}^3$ *is the unit outward normal vector to* Γ_1^θ, *and there exist* $\mathbf{u}_2 \in \mathbf{L}^2(\Omega)$ *and* $\phi_0 \in V^\phi(\Omega)$, $\psi_0 \in V^\psi(\Omega)$, *such that*

$$(\rho\mathbf{u}_2,\mathbf{v})_{\mathbf{L}^2(\Omega)} + c(\mathbf{u}_0,\mathbf{v}) + \varepsilon(\phi_0,\mathbf{v}) + b(\psi_0,\mathbf{v}) - \lambda(\theta_0,\mathbf{v}) = \int_\Omega \sum_{i=1}^{3} f_i(x,0)v_i dx + \int_{\Gamma_1} \sum_{i=1}^{3} g_i(x,0) tr_{\Gamma_1}(v_i) d\Gamma,$$

$$-\varepsilon(\bar{\phi},\mathbf{u}_0)+d(\phi_0,\bar{\phi})+a(\psi_0,\bar{\phi})-\mu(\theta_0,\bar{\phi})=\int_\Omega f^\varepsilon(x,0)\bar{\phi}dx-\int_{\Gamma_1^\phi} g^\phi(x,0)tr_{\Gamma_1^\phi}(\bar{\phi})d\Gamma,$$

$$-b(\bar{\psi},\mathbf{u}_0)+a(\phi_0,\bar{\psi})+\zeta(\psi_0,\bar{\psi})-m(\theta_0,\bar{\psi})=-\int_{\Gamma_1^\psi} g^\psi(x,0)tr_{\Gamma_1^\psi}(\bar{\psi})d\Gamma,$$

for all $\mathbf{v}\in\mathbf{V}(\Omega)$, $\bar{\phi}\in V^\phi(\Omega)$, $\bar{\psi}\in V^\psi(\Omega)$, *then the initial-boundary value problem* (11)-(15) *possesses a unique solution* $(\mathbf{u},\phi,\psi,\theta)$, $\mathbf{u}\in C([0,T];\mathbf{V}(\Omega))$, $\mathbf{u}'\in L^\infty(0,T;\mathbf{V}(\Omega))$, $\mathbf{u}''\in L^\infty(0,T;\mathbf{L}^2(\Omega))$, $\phi\in C([0,T];V^\phi(\Omega)/R_\phi)$, $\phi'\in L^\infty(0,T;V^\phi(\Omega)/R_\phi)$, $\psi\in C([0,T];V^\psi(\Omega)/R_\psi)$, $\psi'\in L^\infty(0,T;V^\psi(\Omega)/R_\psi)$, $\theta\in C([0,T];V^\theta(\Omega))$, $\theta'\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;V^\theta(\Omega))$, *and the mapping*

$$(\mathbf{u}_0,\mathbf{u}_1,\theta_0,\mathbf{f},\mathbf{g},\mathbf{g}',(f^\varepsilon)',(g^\phi)',(g^\psi)',f^\theta,g^\theta)\to(\mathbf{u},\mathbf{u}',\phi,\psi,\theta)$$

is linear and continuous from the space

$$\mathbf{V}(\Omega)\times\mathbf{L}^2(\Omega)\times L^2(\Omega)\times L^2(0,T;\mathbf{L}^2(\Omega))\times L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))\times L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))\times L^2(0,T;L^{6/5}(\Omega))$$
$$\times L^2(0,T;L^{4/3}(\Gamma_1^\phi))\times L^2(0,T;L^{4/3}(\Gamma_1^\psi))\times L^2(0,T;L^{6/5}(\Omega))\times L^2(0,T;L^{4/3}(\Gamma_1^\theta))$$

to the space

$$C([0,T];\mathbf{V}(\Omega))\times C([0,T];\mathbf{L}^2(\Omega))\times C([0,T];V^\phi(\Omega)/R_\phi)\times C([0,T];V^\psi(\Omega)/R_\psi)\times C([0,T];L^2(\Omega)).$$

Remark. Note that if $\Gamma_0^\phi\neq\varnothing$, $\Gamma_0^\psi\neq\varnothing$, then $R_\phi=\{0\}$, $R_\psi=\{0\}$. Consequently, in the case of mixed boundary conditions from Theorem we have that the initial-boundary problem (11)-(15) has a unique solution in the corresponding spaces of vector-valued distributions with values in the subspaces of the first order Sobolev spaces, because $V^\phi(\Omega)/R_\phi=V^\phi(\Omega)$ and $V^\psi(\Omega)/R_\psi=V^\psi(\Omega)$.

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(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

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