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Mathematics

On the Possible Values of Upper and Lower Derivatives with Respect to Differentiation Bases of Product Structure

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ABSTRACT. A solution of the Guzmán's problem on possible values of upper and lower derivatives is given for the class of translation invariant and product type differentiation bases formed by *n*-dimensional intervals. Namely, the bases from the mentioned class are characterized, for which integral means of a summable function can boundedly diverge only on a set of zero measure. *© 2018 Bull. Georg. Natl. Acad. Sci.*

Key words: differentiation of integrals, translation invariant basis, product of bases, upper and lower derivatives

1. Definitions and notation. A family B of open bounded and non-empty subsets of $\mathbb{R}^n$ is called a differentiation basis (briefly: basis) if for every $x \in \mathbb{R}^n$ there exists a sequence (R_k) of sets from B such that $x \in R_k$ $(k \in \mathbb{N})$ and $\lim\limits_{k\to\infty} \operatorname{diam} R_k = 0$.

For a basis B the family of all sets from B containing the point x will be denoted by $B(x)$.

Let B be a basis. For $f \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the upper and lower limits of the integral means $\frac{1}{|R|}\int_R f$, where R is an arbitrary set from $B(x)$ and $\operatorname{diam} R \to 0$, are called the upper and the lower derivatives with respect to B of the integral of f at the point x, and are denoted by $\bar{D}_B(\int f, x)$ and $\underline{D}_B(\int f, x)$, respectively. If the upper and the lower derivatives coincide, then their combined value is called the derivative of $\int f$ at the point x and is denoted by $D_B(\int f, x)$. We say that B differentiates $\int f$ (or $\int f$ is differentiable with respect to B) if $\bar{D}_B(\int f, x) = \underline{D}_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^n$. If this is true for each f in a class of functions $F \subset L(\mathbb{R}^n)$ we say that B differentiates F.

Denote by $\mathbf{I}=\mathbf{I}(\mathbb{R}^n)$ the basis consisting of all n-dimensional intervals. Differentiation with respect to $\mathbf{I}$ is called the strong differentiation.

A basis B is called:

- translation invariant if for every $R\in B$ and a translation $T:\mathbb{R}^n\to\mathbb{R}^n$ we have $T(R)\in B$;
- homothety invariant if for every $R\in B$ and a homothety $H:\mathbb{R}^n\to\mathbb{R}^n$ we have $H(R)\in B$;
- formed by sets from a class Δ if $B\subset\Delta$;
- convex if it is formed by convex sets;
- density basis if B differentiates the integral of the characteristic function of an arbitrary measurable bounded set $E\subset\mathbb{R}^n$.

Note that each homothety invariant basis is translation invariant also.

In what follows the dimension of the space $\mathbb{R}^n$ is assumed to be greater than 1.

If B is a translation invariant and convex basis then for every function $f\in L(\mathbb{R}^n)$ we have

$$\underline{D}_B(\int f,x)\le f(x)\le\overline{D}_B(\int f,x)$$

for almost every $x\in\mathbb{R}^n$. Indeed, there exists a translation invariant basis $B'\subset B$ for which $B'(0)=\{R_k\}$, where $R_1\supset R_2\supset\cdots$ and $\operatorname{diam}R_k\to 0$. For the basis B' it is valid Vitali type covering theorem (see [1, p. 25]). It implies that B' differentiates $L(\mathbb{R}^n)$. Consequently, for every $f\in L(\mathbb{R}^n)$, the estimations $\underline{D}_B(\int f,x)\le\underline{D}_{B'}(\int f,x)=f(x)=\overline{D}_{B'}(\int f,x)\le\overline{D}_B(\int f,x)$ hold almost everywhere.

Let $B_1,\dots,B_k$ be bases in $\mathbb{R}^{n_1},\dots,\mathbb{R}^{n_k}$, respectively. The product basis $B_1\times\cdots\times B_k$ is defined as the basis in $\mathbb{R}^{n_1+\cdots+n_k}$ which consists of all sets $R_1\times\cdots\times R_k$ where $R_1\in B_1,\dots,R_k\in B_k$.

Obviously, the basis $\mathbf{I}(\mathbb{R}^n)$ is the product of n copies of the basis $\mathbf{I}(\mathbb{R})$.

The properties of bases with product structure were studied by various authors (see, e.g., [2-4]).

Let us say that a collection of sets $\{R_1,\dots,R_m\}$ is a partition of a set R if $R=\bigcup_{i=1}^{m}R_i$ and $|R_i\cap R_j|=0$ $(i\ne j)$.

Let us call a basis B complete if there are numbers $0<c_1\le c_2<1$ with the following property: For every set $R\in B$ there exist sets $R_1,\dots,R_m\in B$ composing a partition of R and such that $c_1\le|R_i|/|R|\le c_2$ $(i\in\overline{1,m})$.

2. Result. Besicovitch [5] proved that the strong integral means of a summable function of two variables can boundedly diverge only on a set of zero measure, moreover, for every $f\in L(\mathbb{R}^2)$, both sets

$$\left\{-\infty<\underline{D}_{\mathbf{I}(\mathbb{R}^2)}(\int f,\cdot)<f\right\},\quad\left\{f<\overline{D}_{\mathbf{I}(\mathbb{R}^2)}(\int f,\cdot)<\infty\right\}$$

have zero measure. An analog of this result for the multi-dimensional case was obtained by Ward [6].

We say that a basis B possesses the Besicovitch property if the sets

$$\left\{-\infty<\underline{D}_B(\int f,\cdot)<f\right\},\qquad\left\{f<\overline{D}_B(\int f,\cdot)<\infty\right\}$$

have zero measure for every $f\in L(\mathbb{R}^n)$.

Guzmán [7, p. 389] posed the following problem: To what bases can Besicovitch's result be extended, i.e., what kind of bases the Besicovitch property can possess.

The following theorem gives a solution of the problem for the class of translation invariant and product type differentiation bases formed by n-dimensional intervals.

Theorem 1. *Let* $B = B_1 \times \cdots \times B_n$ *where* $B_1, \ldots, B_n$ *are translation invariant bases formed by one-dimensional intervals. Then the following statements are equivalent:*

1. *the basis* B *possesses the Besicovitch property;*
2. *each among the bases* $B_1, \ldots, B_n$ *is complete.*

The implication $2) \Rightarrow 1)$ was proved in [8]. The converse implication we obtain from the following result.

Theorem 2. *Let* $B = B_1 \times \cdots \times B_n$ *where* $B_1, \ldots, B_n$ *are translation invariant bases formed by one-dimensional intervals. If some among the bases* $B_1, \ldots, B_n$ *is not complete then there exists a non-negative function* $f \in L(\mathbb{R}^n)$ *for which*

$$f(x) < \bar{D}_B(\int f, x) < \infty$$

almost everywhere.

Remark 1. The first example of a translation invariant basis formed by n-dimensional intervals which does not possess the Besicovitch property was constructed in [9]. Note that the basis B constructed in [9] is not of product structure.

Remark 2. A solution of the Guzmán's problem for the class of translation invariant bases formed by n-dimensional intervals is unknown. Moreover, the problem is open even for homothety invariant bases formed by n-dimensional intervals. Note that the last type bases B possess so called weak Besicovitch property: for each non-negative function $f \in L(\mathbb{R}^n)$ both sets $\{-\infty < \underline{D}_B(\int f, \cdot) < f\}$, $\{f < \bar{D}_B(\int f, \cdot) < \infty\}$ have zero measure. This follows from a result of Guzmán and Menárguez [1, p.106] according to which every homothety invariant and density basis formed by central-symmetric convex sets possesses the weak Besicovitch property. Generalization of this result is given in [10].

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