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*Mathematics*

# Inductive Theorems and the Structure of Projective Modules over Crossed Group Rings

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**ABSTRACT. It is proved that the Grothendieck functor and the Swan-Gersten higher algebraic *K*-functors of a crossed group ring $R[\pi,\sigma,\rho]$ are Frobenius modules. As the corollaries an induction theorem for this functors and a reduction theorem for finitely generated $R[\pi,\sigma,\rho]$-projective modules (if *R* is a discrete normalization ring) are proved. Under some restrictions on $n=(\pi:1)$ it is shown that finitely generated $R[\pi,\sigma,\rho]$-projective modules are decomposed into the direct sum of left ideals of the ring $R[\pi,\sigma,\rho]$. More stronger results are proved, when $\sigma=id$.** *© 2018 Bull. Georg. Natl. Acad. Sci.*

**Key words**: crossed group ring, Frobenius modules, induction theorem, algebraic K-functors

In 1960 R. G. Swan proved [1] that a Grothendieck functor $G_0^R(R[\pi])$ of a group ring $R[\pi]$ is a Frobenius functor. As a consequence, he proved that for a Dedekind domain *R* of characteristic 0 and a finite group $\pi$ any finitely generated projective $R[\pi]$-module decomposes as a direct sum of left ideals of $R[\pi]$, if no prime divider of $\pi$ is invertible in *R*. He also proved that this direct sum may be replaced by the direct sum of a free $R[\pi]$ module and an ideal of $R[\pi]$. Swan's results were based on two theorems having an independent value: on the induction theorem for the functors $G_0^R(R\pi)$ and $K_0(R[\pi])$ and on the "reduction" theorem. In 1968 T.Y. Lam [2] proved that $K_1(R[\pi])$ functor is a Frobenius module over $G_0^R(R[\pi])$ and that an induction theorem is valid for $K_1(R[\pi])$. In 1973 A.I. Nemytov [3] proved that Swan-Gersten algebraic *K*-functors $K_n(R[\pi])$, $n\geq 2$ are Frobenius modules over $G_0^R(R[\pi])$ and induction theorems are valid for these functors ([4, 5]). Induction theorems for some kinds of algebraic *K*-functors of group rings were obtained in 1986 by K. Kawakubo [6] and in 2005 [7] by A. Bartels and W. Luck [7].

In this paper we prove (Theorem1) that Swan-Gersten algebraic *K*-functors $K_m(R[\pi,\sigma,\rho])$ are Frobenius modules and generalize an induction theorem for this functors (Theorem 2), where $R[\pi,\sigma,\rho]$ is a crossed group ring. With the help of induction theorem for $K_0(R[\pi,\sigma,\rho])$ a "reduction" theorem for finitely generated projective $R[\pi,\sigma,\rho]$-modules is proved, if *R* is a discrete valuation ring (Theorem 3) and the theorems on the structure of finitely generated projective $R[\pi,\sigma,\rho]$ - and $R[\pi,\rho]$ -modules are obtained, which generalize the above mentioned Swan's theorems.

Let *R* be a commutative ring with identity, $\pi$ a group, $\sigma:\pi\to AutR$ a morphism of groups, $U(R)$ a set of invertible elements of *R* and $\rho:\pi\times\pi\to U(R)$ such a mapping that $\rho(x,y)\rho(xy,z)=\rho(y,z)^x\rho(x,yz)$. Then a crossed group ring $R[\pi,\sigma,\rho]$ ([8, 9]) is a free *R*-module with the set of free generates $\pi$ and with multiplication $r_1\overline{x_1}r_2\overline{x_2}=r_1r_2^{x_1}\rho(x_1,x_2)\overline{x_1x_2}$, where $\overline{x}$ is an image of $x\in\pi$ via a mapping $\pi\to R[\pi,\sigma,\rho]$ and $r_1,r_2\in R$. If $\sigma(\pi)=id$ and $\rho\sim1$ (i.e. $\rho(x,y)=\alpha(x)\alpha(y)\alpha(xy)^{-1}$, $\alpha:\pi\to U(R)$), then $R[\pi,\sigma,\rho]\cong R[\pi]$. Further all modules are the left modules, $\underline{M}(A)$ and $\underline{P}(A)$ denotes respectively categories of finitely generated *A*-modules and finitely generated projective *A*-modules; $\underline{M}^R(R[\pi,\sigma,\rho])$ is a category of finitely generated *R*-projective $R[\pi,\sigma,\rho]$ -modules; $G_0^R(R[\pi,\sigma,\rho])$ is a Grothendieck group of the category $\underline{M}^R(R[\pi,\sigma,\rho])$ and $\pi$ will be always a finite group.

Main results of the paper are Theorems 6 and 7. These theorems were proved by the author in [10-12]. The particular case when $\rho\sim1$ was announced in [12] and its proof was a subject of the author's doctoral thesis in 1981. This theorems are similar to the results of Kawakubo [6], which were obtained later in 1986 for some kinds of algebraic K-functors of group rings and particular cases of crossed group rings.

Let $\underline{G}$ be a category, $\underline{Rings}$ - a category of rings, $G:\underline{G}\to\underline{Rings}$ - a contravariant functor, $i^*=G(i):G(\pi)\to G(\pi')$. If to each morphism $i:\pi'\to\pi$ in $\underline{G}$ corresponds such a morphism $i_*:G(\pi')\to G(\pi)$ in *Rings* that $Id_*=Id$ and $(ij)_*=i_*j_*$ (whenever $ij$ makes sense), then a functor *G* is called a Frobenius functor [2] if it satisfies the Frobenius reciprocity formula $i_*(i^*a\cdot b)=a\cdot i_*b$.

Let $\underline{Ab}$ be a category of commutative groups. A contravariant functor $K:\underline{G}\to\underline{Ab}$ is called a Frobenius module [2] on the Frobenius Functor *G* if it satisfies the following conditions: (i) $K(\pi)$ is a module over $G(\pi)$; (ii) For each morphism of groups $i:\pi'\to\pi$ a morphism $i_\#:K(\pi')\to K(\pi)$ exists (whenever $ij$ makes sense) that $(ij)_\#=i_\#j_\#$; (iii) $i_\#(y\cdot i^\#(a))=i_*(y)\cdot a$, $i_\#(i^*(\mathrm{x})\cdot\mathrm{b})=x\cdot i_\#(\mathrm{b})$. Here $i^\#=K(i)$, $x\in G(\pi)$, $y\in G(\pi')$, $a\in K(\pi)$, $b\in G(\pi')$.

Let $\underline{G}(\pi)$ denote a category whose objects are all subgroups $\pi'\subseteq\pi$ and morphisms – monomorphisms $i:\pi'\to\pi$. Then the functors $G_0^S(S[-])$ and $K_m(R[-,\sigma,\rho])$ are contravariant functors from the category $\underline{G}(\pi)$ to the categories $\underline{Rings}$ and $\underline{Ab}$, respectively. It is known [1] that $G_0^S(S[\pi])$ is a Frobenius functor.

Let us denote $R^\pi=\{r\in R\,|\,(\forall x\in\pi)r^x=r\}$.

**Theorem 1**. *Let $R^\pi$ be an algebra over a commutative ring S with identity. Then $G_0^R(R[-,\sigma,\rho])$ and $K_m(R[-,\sigma,\rho])$, $m=0,1,\ldots$ functors are Frobenius modules on the Frobenius functor $G_0^S(S[\pi])$.*

If $R^\pi$ is an algebra over *S*, then *R* is a *S*-algebra. Let us construct a morphism of rings $\alpha:R[\pi,\sigma,\rho]\to S[\pi]\otimes_S R[\pi,\sigma,\rho]$, $\alpha(r\overline{x})=\overline{x}\otimes r\overline{x}$. Then for any $S[\pi]$-module *M* and $R[\pi,\sigma,\rho]$-module *P* the module $M\otimes_S P$ is a $R[\pi,\sigma,\rho]$ -module by the action $r\overline{x}(m\otimes p)=\alpha(r\overline{x})(m\otimes p)=\overline{x}m\otimes r\overline{x}p$. Let us denote such a module by $\langle M\otimes_S P\rangle$

**Proposition 1**. *If $S[\pi]$-module M is S-projective and $R[\pi,\sigma,\rho]$-module P is $R[\pi,\sigma,\rho]$-projective, then $\langle M\otimes_S P\rangle$ is $R[\pi,\sigma,\rho]$-projective.*

**Proposition 2**. *Let $R^\pi$ be an algebra over S, $\pi'\subseteq\pi$ - a subgroup, $M\in S\pi-\underline{Mod}$, $M'\in S\pi'-\underline{Mod}$, $P\in R[\pi,\sigma,\rho]-\underline{Mod}$ and $P'\in R[\pi',\sigma,\rho]-\underline{Mod}$. Then*

$$R[\pi,\sigma,\rho]\otimes_{R[\pi',\sigma,\rho]}\langle M'\otimes_S P\rangle\cong\left\langle(R[\pi,\sigma,\rho]\otimes_{R[\pi',\sigma,\rho]}M')\otimes_S P\right\rangle$$

as $R[\pi,\sigma,\rho]$ *-modules.*

Theorem 1 follows from Propositions 1 and 2 and results from [13].

Let $M$ be some family of objects from $\underline{G}$. Let us denote for $\pi \in \underline{G}$

$$K_\pi(M) = \sum_{\pi',i} \{\mathrm{Im}(i_\# : K(\pi') \to K(\pi)) \mid i : \pi' \to \pi, \pi' \in \mathrm{M}\}.$$

Let $A \subseteq B$ be abelian groups and $n$ an index of $A$ in $B$, i.e. $nB \subseteq A$. From Theorem 1 follows the **induction** theorem:

**Theorem 2**. *Let* $c(\pi)$ *be a set of all cyclic subgroups of group* $\pi$. *Then* $K_m(R[\pi,\sigma,\rho])_{c(\pi)}$ *and* $G_0^R(R[\pi,\sigma,\rho])_{c(\pi)}$ *have an index* $n^2$ *in* $K_m(R[\pi,\sigma,\rho])$ *and* $G_0^R(R[\pi,\sigma,\rho])$ *respectively for all* $m \ge 0$. *If* $R^\pi$ *is an algebra over a field, then* $n^2$ *may be replaced by n.*

From the theorems above follows the **reduction** theorem for $R[\pi,\sigma,\rho]$ -projective modules*:*

**Theorem 3**. *Let* $R$ *be a discrete valued ring with the quotient field K;* $P,Q \in \underline{P}(R[\pi,\sigma,\rho])$ *and* $K \otimes_R P \cong K \otimes_R Q$ *as* $K[\pi,\sigma,\rho]$ *-modules. Then* $P \cong Q$ *as* $R[\pi,\sigma,\rho]$ *-modules.*

**Remark**. $K[\pi,\sigma,\rho]$ *acts on* $K \otimes_R P$ *as* $\bar{x}(\alpha \otimes p) = \alpha^x \otimes xp$.

This theorem was proved by Swan [1] for group rings.

To prove Theorem 3 it suffices to prove the following

**Theorem 4**. *Let k be a field. Then Cartan homomorphis* $\chi : K_0(k[\pi,\sigma,\rho]) \to G_0(k[\pi,\sigma,\rho])$ *is injective.*

Theorem 4 itself reduces to the case when the group is cyclic; for cyclic groups Theorem 4 follows from

**Theorem 5**. *Let A be a (noncommutative) principal ideal domain, in which each ideal is bounded. Let* $I \subseteq A$ *two sided ideal,* $K_0(A/I)$ *and* $G_0(A/I)$ *- Grothendieck groups of the categories* $\underline{P}(A/I)$ *and* $\underline{M}(A/I)$ *respectively. Then Cartan homomorphism* $\chi : K_0(A/I) \to G_0(A/I)$ *is injective.*

Now we can study the projective $R[\pi,\sigma,\rho]$ -modules if $R$ is a Dedekind domain.

Let $\omega = Ker(\sigma : \pi \to Aut(R))$. If $\sigma(\pi) = id$, we denote $R[\pi,\sigma,\rho] := R[\pi,\rho]$.

**Theorem 6**. *Let R a Dedekind domain* $charR = 0$. *Suppose no one prime divider of* $n$ *is invertible in R and (i) R is* $R^\pi$ *-projective; (ii) if* $\mathbf{p} \in spec(R)$, $\mathbf{p} \mid (n)$, *then* $\sigma(\pi)(\mathbf{p}) \subseteq \mathbf{p}$ *; (iii) if* $p$ *is a prime divider of the number* $n$, $p \in \mathbf{p} \in spec(R)$ *and* $\pi_p$ *is a Sylov p- subgroup of* $\pi$, *then* $\pi_p$ *acts trivially on* $R/\mathbf{p}$ *; (iv)* $\rho(\pi \times \pi) \subseteq R^\pi$. *Then any finitely generated projective* $R[\pi,\sigma,\rho]$ *-module splits in direct sum of left ideals of the ring* $R[\pi,\sigma,\rho]$.

In a particular case when $\sigma(\pi) = id$, we may prove a stronger result.

**Theorem 7**. *Let R be a Dedekind domain* $charR = 0$. *If no one prime divider of* $n = (\pi : 1)$ *is invertible in R, then any finitely generated projective* $R[\pi,\rho]$ *-module is the direct sum of a free* $R[\pi,\rho]$ *-module and a left ideal* $I \subseteq R[\pi,\rho]$. *For any non-zero ideal* $\mathbf{j} \subseteq R$ *an ideal* $I$ *can be chosen in such a way that* $I$ *and* $\mathbf{j}$ *would be coprime ideals.*

**Proof of Theorem 6**. It exist such an imbedding of a module $P$ in a free $R[\pi,\sigma,\rho]$ -module $F$ that $(P:F)+(n) = R$, $(P:F)_{R^\pi} + nR^\pi = R^\pi$. Let $a_1, a_2, ..., a_k$ be $R[\pi,\sigma,\rho]$ -basis of $F$. Let us consider a morphism of $R[\pi,\sigma,\rho]$ -modules $\varphi_1 : F \to R[\pi,\sigma,\rho]$, $\sum_i \mu_i a_i \to \mu_1$. Since $rF \subseteq P \Rightarrow rR[\pi,\sigma,\rho] \subseteq I_1$, thus $(P:F) \subseteq (I_1 : R[\pi,\sigma,\rho])$. Therefore from $(P:F)+(n) = R$ follows that $(I_1 : R[\pi,\sigma,\rho]) + (n) = R$. Then the ideal $I_1$ is $R[\pi,\sigma,\rho]$ -projective. $\varphi : P \to I_1$ is surjective, therefore $P \cong P' \oplus I_1$. The theorem is easy to prove by mathematical induction on $rk_K(P)$.

**Proof of Theorem 7**. Under the conditions of Theorem 7 any module $P$ is isomorphic to a direct sum $\sum I_i$ of ideals of $R[\pi,\rho]$; in addition for any nonzero ideal $J \subseteq R$ the ideals $I_i$ can be chosen in such a way that $(I_i : R[\pi,\rho]) + J = R$ for all $i$. We may suppose $K \otimes_R I_i \cong K[\pi,\rho]$. Then it is sufficient to prove the following: let $I_1, I_2 \subseteq R[\pi,\rho]$ be such a projective ideals that $(I_1 : R[\pi,\rho])$ and $(I_2 : R[\pi,\rho])$ are coprime to $J$ and $K \otimes_R I_1 \cong K \otimes_R I_2 \cong K[\pi,\rho]$; then $I_1 \oplus I_2 \cong R[\pi,\rho] \oplus I$, where $I \subseteq R[\pi,\rho]$ is a left ideal and $(I : R[\pi,\rho]) + J = R$.

Let $J_1 = (I_1 : R[\pi,\rho])$. It exist $I_2' \subseteq R[\pi,\rho]$ such that $I_2 \cong I_2'$ and $(I_2' : R[\pi,\rho]) + JJ_1 = R$. Let us replace $I_2$ with $I_2'$. Therefore, we may assume that there exist $b_1 \in (I_1 : R[\pi,\rho])$ and $b_2 \in (I_2 : R[\pi,\rho])$ such that $b_1 + b_2 = 1$. Let $F$ be the free $R[\pi,\rho]$-module with two free generators $e_1, e_2$ and $V = I_1 e_1 + I_2 e_2 \subseteq F$. Then $A \cong I_1 + I_2$ and $(V : F) + J = R$. It is clear that the elements $e_1' = e_1 b_1 + e_2 b_2$ and $e_2' = e_1 - e_2$ are also free generators of $F$, because $e_1 = e_1' + b_2 e_2'$, $e_2 = e_1' - b_2 e_2'$. But $e_1' \in V$ because $b_1 \in I_1$, $b_2 \in I_2$. Consequently $V = R[\pi,\rho] e_1' + I e_2'$ where $I = \{a \in R[\pi,\rho] \mid r e_2' \in V\}$. It is clear also that $(I : R[\pi,\rho]) + J = R$ because $(I : R[\pi,\rho]) = (V : F)$.

*მათემატიკა*

# ინდუქციური თეორემები და ჯვარედინ ჯგუფურ რგოლზე პროექციული მოდულების სტრუქტურა

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დამტკიცებულია, რომ $R[\pi,\sigma,\rho]$ ჯვარედინი ჯგუფური რგოლის გროტენდიკის ფუნქტორი და სუონ-გერსტენის უმაღლესი *K*-ფუნქტორები ფრობენიუსის მოდულებია. შედეგად დამტკიცებულია ამ ფუნქტორებისათვის ინდუქციურობის და რედუქციულობის თეორემები სასრულად წარმოქმნილი $R[\pi,\sigma,\rho]$-პროექციული მოდულებისათვის, როცა ***R*** არის დისკრეტულად ნორმირებული რგოლი. როდესაც ***R*** არის დედეკინდის რგოლი, $\pi$ ჯგუფის რიგზე გარკვეული შეზღუდვებით დამტკიცებულია, რომ სასრულად წარმოქმნილი $R[\pi,\sigma,\rho]$-პროექციული მოდულები იშლება $R[\pi,\sigma,\rho]$ რგოლის მარცხენა იდეალების პირდაპირ ჯამად. უფრო ძლიერი შედეგებია მიღებული, როდესაც $\sigma = id$.

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