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Mathematics

On Approximate Solution of a Nonlinear Static Beam Equation

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ABSTRACT. In the present paper we consider the approximate solution issues for nonlinear boundary value problem for the Kirchhoff type static beam. The problem is reduced by means of Green's function to a nonlinear integral equation. To solve this problem we use the Picard type iterative method. For both of these problems the new algorithms of approximate solutions are constructed and numerical experiments are executed. The results of computations are presented in tables and diagrams. © *2018 Bull. Georg. Natl. Acad. Sci.*

Key words: Kirchhoff type static beam, nonlinear integal equation, Green's function, Picard iteration method, numerical realization

Statement of the problem. Let us consider the nonlinear beam equation

$$u''''(x)-m\left(\int_0^\ell u'^2(x)dx\right)u''(x)=f\left(x,u(x),u'(x)\right),\quad 0<x<\ell, \tag{1}$$

with the conditions

$$u(0)=u(\ell)=0,\ u''(0)=u''(\ell)=0, \tag{2}$$

Here $u=u(x)$ is the displacement function of length l of the beam subjected to the action of a force given by the function $f\left(x,u(x),u'(x)\right)$, the function $m(z)$,

$$m(z)\geq\alpha>0,\qquad 0\leq z<\infty, \tag{3}$$

describes the type of a relation between stress and strain. Namely, if the function $m(z)$ is linear, this means that this relation is consistent with Hooke's linear law, while otherwise we deal with material nonlinearites.

The questions of the solvability of problem (1), (2) is studied in [1]-[2], while the problem of construction of numerical algorithms and estimation of their accuracy is investigated in [1]-[5]. In the present paper, in order to obtain an approximate solution of the problem (1), (2) an approach is used, which differs from those applied in the above-mentioned references. It consists in reducing the problem (1), (2) by means of Green's function to a nonlinear integral equation to solve it we use the iterative process. The condition for the convergence of the method is estabilished and numerical realization is obtained.

Remark: For the solution and investigation of the abovementioned problem we use certain assumptions and limitations, which are foreseen for the following two functions $m(z)$ and $f(x,u,v)$, also for the length of beam ℓ (see [5], [6]).

The method. Let us assume that there exists a solution of the problem (1), (2) and $u(x) \in W_0^{2,2}(0,\ell)$.

We will need the Green function for the problem

$$\begin{aligned} &v''''(x) - av''(x) = \psi(x), \\ &0 < x < \ell, \ a = const > 0, \\ &v(x) = v(\ell) = 0, \ v''(0) = v''(\ell) = 0. \end{aligned} \tag{4}$$

In order to obtain this function, we split problem (4) into two problems

$$\begin{aligned} &w''(x) - aw(x) = \psi(x), \\ &w(0) = w(\ell) = 0, \end{aligned}$$

and

$$\begin{aligned} &v''(x) = w(x), \\ &v(0) = v(\ell) = 0. \end{aligned}$$

Calculations convince us that

$$w(x) = -\frac{1}{\sqrt{a}\sinh\left(\sqrt{a}\ell\right)}\left(\int_0^x \cosh\left(\sqrt{a}(x-\ell)\right)\cosh\left(\sqrt{a}\xi\right)\psi(\xi)\,d\xi\right) -$$

$$-\frac{1}{\sqrt{a}\sinh\left(\sqrt{a}\ell\right)}\left(\int_x^\ell \cosh\left(\sqrt{a}(\xi-\ell)\right)\cosh\left(\sqrt{a}x\right)\psi(\xi)\,d\xi\right),$$

$$v(x) = \frac{1}{\ell}\left(\int_0^x (x-\ell)\xi w(\xi)\,d\xi + \int_x^\ell x(\xi-\ell)w(\xi)\,d\xi\right).$$

Substituting the first of these formulas into the second and performing integration by parts, we obtain

$$v(x) = \frac{1}{a}\int_0^x \left(\frac{1}{\ell}(\ell-x)\xi + \frac{1}{\sqrt{a}\sinh\left(\sqrt{a}\ell\right)}\sinh\left(\sqrt{\tau}(x-\ell)\right)\sinh\left(\sqrt{\tau}\xi\right)\psi(\xi)\right)d\xi +$$

$$+\frac{1}{a}\int_x^\ell \left(\frac{1}{\ell}x(\ell-\xi) + \frac{1}{\sqrt{a}\sinh\left(\sqrt{a}\ell\right)}\sinh\left(\sqrt{\tau}(\xi-\ell)\right)\sinh\left(\sqrt{\tau}x\right)\psi(\xi)\right)d\xi. \tag{5}$$

The application of (4), (5) to problem (1), (2) makes it possible to replace the latter problem by the integral equation

$$u(x) = \int_0^\ell G(x,\xi)f\left(\xi,u(\xi),u'(\xi)\right)d\xi, \quad 0 < x < \ell, \tag{6}$$

where

$$G(x,\xi)=\frac{1}{\tau}\begin{cases}\frac{1}{\ell}(\ell-x)\xi+\frac{1}{\sqrt{\tau}\sinh\left(\sqrt{\tau}\ell\right)}\sinh\left(\sqrt{\tau}(x-\ell)\right)\sinh\left(\sqrt{\tau}\xi\right), & 0<\xi\le x<\ell,\\ \frac{1}{\ell}x(\ell-\xi)+\frac{1}{\sqrt{\tau}\sinh\left(\sqrt{\tau}\ell\right)}\sinh\left(\sqrt{\tau}(\xi-\ell)\right)\sinh\left(\sqrt{\tau}x\right), & 0<x\le\xi<\ell,\end{cases}$$

$$\tau=m\left(\int_0^\ell u'^2(x)dx\right).$$

The equation (6) is solved by the method of the Picard iterations. After choosing a function $u_0(x), 0\le x\le\ell,$ which together with its second derivative vanish for the $x=0$ an $x=\ell$, we find subsequent approximations by the formula

$$u_{k+1}(x)=\int_0^\ell G_k(x,\xi)f\left(\xi,u_k(\xi),u_k'(\xi)\right)d\xi, \quad 0<x<\ell, \quad k=0,1,... \tag{7}$$

where

$$G_k(x,\xi)=\frac{1}{\tau_k}\begin{cases}\frac{1}{\ell}(\ell-x)\xi+\frac{1}{\sqrt{\tau_k}\sinh\left(\sqrt{\tau_k}\ell\right)}\sinh\left(\sqrt{\tau_k}(x-\ell)\right)\sinh\left(\sqrt{\tau_k}\xi\right), & 0<\xi\le x<\ell,\\ \frac{1}{\ell}x(\ell-\xi)+\frac{1}{\sqrt{\tau_k}\sinh\left(\sqrt{\tau_k}\ell\right)}\sinh\left(\sqrt{\tau_k}(\xi-\ell)\right)\sinh\left(\sqrt{\tau_k}x\right), & 0<x\le\xi<\ell,\end{cases}$$

$$\tau_k=m\left(\int_0^\ell u_k'^2(x)dx\right),$$

$k=0,1,...$ and $u_k(x)$ is the k-th approximation of the solution of equation (6).

The computer experiments for the corresponding problems are carried out. The issues of experimental convergence are studied.

The numerical realization. For approximate solution of the boundary value problem (1), (2) several programs are composed in "Maple" and many numerical experiments are carried out. The obtained results are good enough. The algorithm is approved with tests and the results of recounts are represented in tables and graphics.

Theoretical results about convergence of approximate solutions for a nonlinear integral equation (6) of sought function $u(x)$ of the nonlinear static beam obtained by iterative method $u_k(x)$ is confirmed by the numerical experiment.

By way of illustration, the results of numerical computations of two the test problems are given in the paper.

The algorithm is approved in the following two tasks on the test and their illustrations in the article are given.

Task 1. We realize the analogical example, which is considered in paper [1]. We consider a special case, where $m(z)=m_0+m_1\cdot z,\ m_0,m_1>0,$ the beam length $\ell=1,$ the right-hand side $f\left(x,u(x),u'(x)\right)=1+\left(u(x)\right)^2+\left(u'(x)\right)^2$. We carried out seven iterations. To compute the integrals in the case of division of the interval into n=20,40,80 parts (mesh h=0.05, 0.025, 0.0125 respectively), the generalized square formula of trapezium [0,1] was used. The issues of convergence of approximate solutions are studied. Numerical values for the maximum modules of differences in two successive iterative

solutions are computed at knots (see Table 1-3). Initial approximation of approximate solution $u_0(x)=0$.
The error in k-iteration:

$$error\,k=\max_{i=0,1,\dots,n}\left\{abs\left(u_k(x_i)-u_{k-1}(x_i)\right)\right\},\ k=1,2,\dots,7.$$

Table 1. Test problem 1., Coefficients $m_0=1,\ m_1=1.$

n\error	error1	error2	error3	error4	error5	error6	error7
n=20	0.2382047	0.0231688	0.0006797	0.0002911	0.0000147	0.0000039	0.00000037
n=40	0.2381870	0.0233660	0.0007256	0.0003075	0.0000130	0.0000045	0.00000035
n=80	0.2381826	0.0234175	0.0007386	0.0003118	0.0000125	0.0000046	0.00000034

Table 2. Test problem 1., Coefficients $m_0=1,\ m_1=0.1.$

n\error	error1	error2	error3	error4	error5	error6	error7
n=20	0.2382047	0.0366517	0.0093673	0.0020780	0.0003816	0.0000569	0.00000069
n=40	0.2381870	0.0368218	0.0093781	0.0020371	0.0003526	0.0000472	0.00000045
n=80	0.2381826	0.0368640	0.0093797	0.0020256	0.0003445	0.0000446	0.00000039

Table 3. Test problem 1., Coefficients $m_0=1,\ m_1=0.01.$

n\error	error1	error2	error3	error4	error5	error6	error7
n=20	0.2382047	0.0444195	0.0161153	0.0060773	0.0023081	0.0008824	0.0003392
n=40	0.2381870	0.0446502	0.0162252	0.0061074	0.0023091	0.0008784	0.0003362
n=80	0.2381826	0.0447079	0.0162526	0.0061146	0.0023091	0.0008772	0.0003354

Task 2. We consider a special case, where $m(z)=m_0+m_1\cdot z,\ m_0,m_1>0,$ the beam length $\ell=1,$ exact solution $u(x)=x(x-1)(x^2-x-1),$ i.e. $u(x)=x^4-2x^3+x,\ m_0=1, m_1=0.5,$ the right-hand side

$$f(x,u(x),u'(x))=\frac{1}{35}\left(\frac{87}{2}u^2(x)-348x^3u'(x)-1566u(x)+696x^6-3132x^3+2088x+\frac{1593}{2}\right).$$

We carried out nine iterations. To compute the integrals in the case of division of the interval into n=10,20 parts (mesh h=0.1, 0.05, respectively), the generalized trapezoidal rule [0,1] was used. The errors in the next two iterations are studied. The error in k-iteration:

$$error\,k=\max_{i=0,1,\dots,n}\left\{abs\left(u_k(x_i)-u_{exact}(x_i)\right)\right\},\ k=1,2,\dots,9.$$

Numerical values for the maximum modules of differences of exact and iterative solutions are computed at knots (see Table 4). Initial approximation of approximate solution $u_0(x)=0.$ In case of five and seven iterations for n=10, 20 the accurate and approximate solutions are graphically illustrated (Figs. 1-4).

Table 4.Test problem 2. Coefficients $m_0=1, m_1=0.5,$ **exact solution** $u(x)=x^4-2x^3+x.$

n\error	error1	error2	error3	error4	error5	error6	error7	error8	error9
n=10	0.43203	0.16734	0.06405	0.02473	0.00953	0.00367	0.00142	0.00055	0.00021
n=20	0.43328	0.16715	0.06365	0.02446	0.00938	0.00360	0.00138	0.00053	0.00020

Remark: in the diagrams (Figs.) the green line/color denotes the accurate solution graph, yellow is the first approximation, red – the second, blue – the third, pink - the fourth, golden - the fifth, brown – the sixth and purple - the seventh.

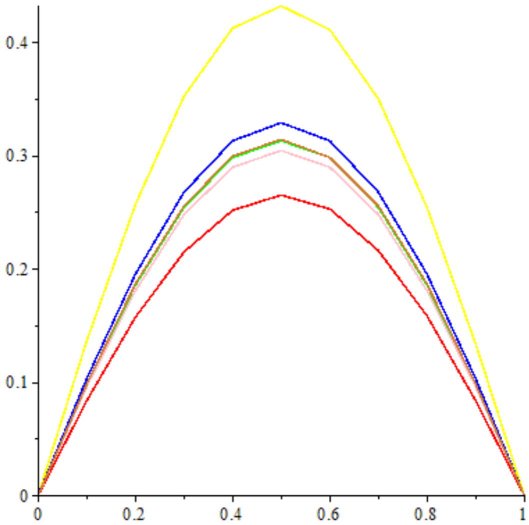

Fig. 1. Iteration=5, n=10.

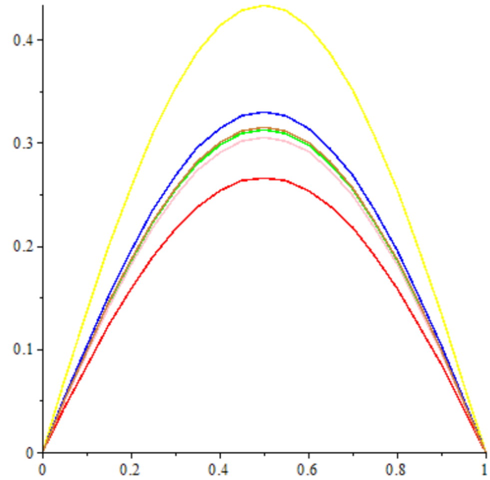

Fig. 2. Iteration=5, n=20.

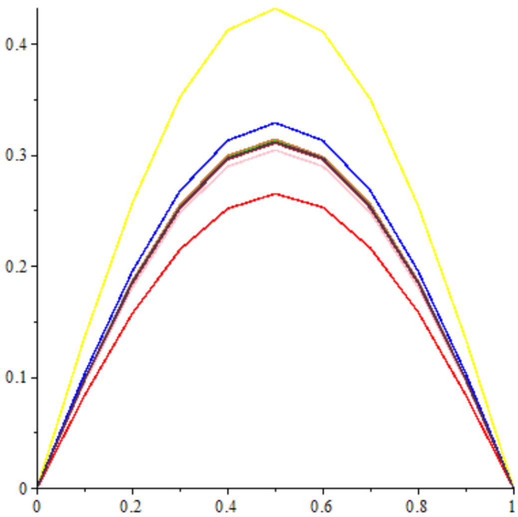

Fig.3. Iteration =7, n=10.

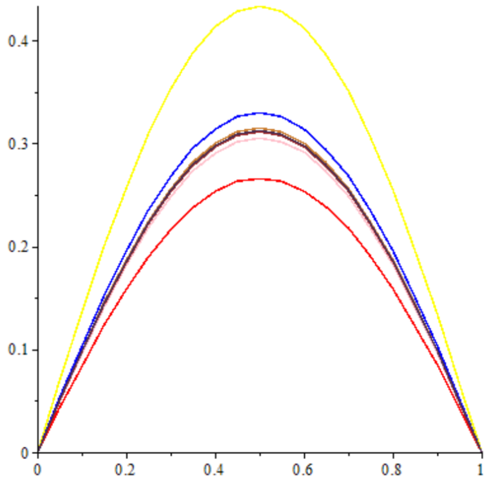

Fig.4. Iteration =7, n=20.

Conclusions.

Numerical experiments clearly show the convergence of approximate solutions with the increase of the number of iterations and the influence of n number of the interval divisions on the rate of convergence, also change of parameters m_0 and m_1 have influence on rapidity of convergence.

მათემატიკა

არაწრფივი სტატიკური ძელის განტოლების მიახლოებითი ამოხსნის შესახებ

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