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Mathematics

Optimality Conditions for *m*-Point Nonlocal Boundary Value Problems

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ABSTRACT. The problem of optimal control with an integral quality criterion is considered for m-point nonlocal boundary value problems and the optimality conditions are obtained. An iterative process is constructed for the investigation of the conjugate problem. *© 2018 Bull. Georg. Natl. Acad. Sci.*

Key words: nonlocal boundary value problem, optimal control

The study of nonlocal boundary value problems, the development and analysis of the methods for their numerical solution is an important direction of applied mathematics. Many papers are devoted to the study of nonlocal boundary value problems (see, for example, [1-7].) Nonlocal boundary value problems for the first order quasilinear differential equations on the plane were considered in the works [8, 9]. An m-point nonlocal boundary value problem for generalized analytic functions is formulated in the works [10, 11], where the investigation is carried out by the method of reducing nonlocal boundary value problems to a sequence of Riemann-Hilbert problems. To the investigation of optimal control problems for nonlocal boundary value problems and numerical methods for their solution are devoted the works [12-16].

The present paper is dedicated to the problems of optimal control whose behavior is described by first order linear differential equations on the plane with m-point nonlocal boundary conditions. Necessary and sufficient conditions of optimality are obtained. A conjugate equation is constructed in the differential form using non-classical boundary conditions. A theorem on the existence and uniqueness of a generalized solution of the conjugate problem is proved. A numerical algorithm of the solution of an optimal control problem is given.

Formulation of the optimal control problem

Let $\overline{G}$ be a rectangle $\{z = x+iy : 0 \le x \le 1, 0 \le y \le 1\}$ of the complex domain E. Γ is the boundary of the domain G and $\gamma_k = \{z = x_k + iy;\ \ 0 \le y \le 1\}$, $k = 1,\dots,m$, $\gamma = \{z = 1+iy : 0 \le y \le 1,\}$, $z^* \in \Gamma \setminus \gamma$,

$z \in G$, $w(z) = w_1(x,y) + iw_2(x,y)$, $\partial_{\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ is a generalized Sobolev derivative [17], $C(\overline{G})$ is the Banach space consisting of all functions continuous on $\overline{G}$. The norm in $C(\overline{G})$ is defined by the equality $\|f\|_{C(\overline{G})} = \max_{z\in\overline{G}} |f(z)|$. $C_\alpha(\overline{G})$ is the set of all bounded functions satisfying the Hölder condition with exponent α. The norm in $C_\alpha(\overline{G})$ is defined by the equality $\|f\|_{C_\alpha(\overline{G})} = \max_{z\in\overline{G}} |f(z)| + \sup_{z_1,z_2\in\overline{G}} \frac{|f(z_1)-f(z_2)|}{|z_1-z_2|^\alpha}$.

$W_p^m(\overline{G})$ is the set of all functions from $L_p(\overline{G})$ which have all types of $D^k f$, $|k| \le m$, are generalized derivatives from the space $L_p(\overline{G})$. The norm in $W_p^m(\overline{G})$ is defined by the equality

$$\|u\|_{W_p^m(G)} = (\int_G \sum_{|k|=0}^{m} |D^k u(x)|^p dx)^{1/p} \equiv (\sum_{|k|=0}^{m} \|D^k u(x)\|_{L_p(G)})^{1/p}.$$

$\overset{0}{W}{}_2^1(G)$ is a subspace of the space $W_2^1(G)$ where the dense set consists of continuously differentiable functions in $\overline{G}$ which vanish on Γ [18].

Let $c(z), d(z), B(z), f(z) \in L_p(\overline{G})$, $2 < p < \frac{1}{1-\alpha}$, $g(z) \in C_\alpha(\Gamma \setminus \gamma)$, $1/2 < \alpha < 1$, U be some subset from E. Each function $\omega(z): G \to U$ will be called a control. The set U is called the domain of control. A function $u(z)$ will be called an admissible control if $\omega(z) \in L_p(G)$. The set of all admissible controls will be denoted by Ω.

For each fixed $\omega \in \Omega$, in the domain $\overline{G}$ we consider the following m-point nonlocal boundary value problem for a system of linear differential equations of the first order:

$$\begin{gathered} \partial_{\bar{z}} w + B(z)\overline{w} = f(z)\omega, \ z \in G, \\ Re[w(z)] = g(z), \ z \in \Gamma \setminus \gamma, \quad Im[w(z^*)] = const, \\ Re[w(z)] = \sum_{k=1}^{m} \sigma_k Re[w(z_k)], \ z \in \gamma, \ z_k \in \gamma_k, \ 0 < \sigma_k = const, \\ \sum_{k=1}^{m} \sigma_k < 1, \ k = 1, \dots, m. \end{gathered} \tag{1}$$

Note that for each fixed $\omega \in \Omega$, problem (1) has a unique solution that belongs to the space $C_\mu(\overline{G})$, $\mu = \frac{p-2}{p}$ [10, 11].

We consider the functions

$$I(\omega) = Re\iint_G [c(z)w(z) + d(z)\omega(z)]\, dxdy \tag{2}$$

and formulate the following optimal control problem: Find a function $\omega_0(z) \in \Omega$ for which the solution of the non-local boundary value problem (1) gives a minimal value to functional (2). The answer to this question is given by the following.

Theorem. *Let $\psi(z)$ be the solution of the conjugate problem*

$$\begin{gathered} \partial_{\bar{z}}\psi(z) - \overline{B}(z)\overline{\psi}(z) = c(z), \ z \in G \setminus \bigcup_{k=1}^{m} \gamma_k, \\ Re[\psi] = 0, \ z \in \Gamma, \\ Re[\psi(z_k^+) - \psi(z_k^-)] = \sigma_k Re[\psi(z)], \ z_k \in \gamma_k, \ z \in \gamma, \ k = 1, \dots, m, \end{gathered} \tag{3}$$

then for the optimality of $\omega_0(z)$, $w_0(z)$ it is necessary and sufficient that the following equality

$$Re[(d(z) - \psi(z) f(z))\omega_0(z)] = \inf_{\omega\in U} Re[(d(z) - \psi(z) f(z))\omega(z)]$$

be fulfilled almost everywhere on G.

Existence and uniqueness of a solution of the conjugate problem

In the rectangle $\overline{G}=[0,1]\times[0,1]$, consider the following boundary value problem

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}+A_1(x,y)\frac{\partial u(x,y)}{\partial x}+A_2(x,y)\frac{\partial u(x,y)}{\partial y}$$

$$+A_3(x,y)u(x,y)=f(x,y),\ (x,y)\in G\setminus\bigcup_{k=1}^{m}\gamma_k,$$

$$u(x,y)=0,\ (x,y)\in\Gamma, \tag{4}$$

$$\frac{\partial u(x_k^+,y)}{\partial x}-\frac{\partial u(x_k^-,y)}{\partial x}=\sigma_k\frac{\partial u(1,y)}{\partial x},\ 0\le y\le 1,\ 0<\sigma_k=const,$$

$$\sum_{k=1}^{m}\sigma_k<1,\ k=1,\ldots,m,$$

where $f(x,y),A_1(x,y),A_2(x,y),A_3(x,y)\in L_p(\overline{G})$, $p>2$, $0\ge A_3(x,y)\in L_\infty(\overline{G})$.

If we introduce the notations

$$\frac{\partial u(x,y)}{\partial x}=\psi_1(x,y),\ \frac{\partial u(x,y)}{\partial y}=-\psi_2(x,y),\ \psi(z)=\psi_1(x,y)+i\psi_2(x,y),$$

$$A_1(x,y)=\frac{1}{2}(B_2(x,y)-B_1(x,y)),\ A_2(x,y)=-\frac{1}{2}(B_1(x,y)+B_2(x,y)),$$

$$f(x,y)=\frac{1}{2}(c_1(x,y)+c_2(x,y)),\ B(z)=B_1(x,y)+iB_2(x,y),$$

$$c(z)=c_1(x,y)+ic_2(x,y),$$

then, in the case $A_3(x,y)=0$, problem (4) is equivalent to the conjugate problem (3).

Let us make a few observations about the properties of the solution of problem (3) [19]. First we observe that if the function $\psi(x,y)=\psi_1(x,y)+\psi_2(x,y)$ is the solution of problem (3), then the solution of problem (3) is obtained by means of the following curvilinear integral

$$u(x,y)=C_0+Re\int_{z_0}^{z}\psi(\zeta)d\zeta,\ C_0=const. \tag{5}$$

Note that for the simply connected domain G the right-hand part of equality (5) for fixed z_0 and C_0 will be a single-valued function of the point z.

The solution of equation (4) is understood in a generalized sense: let $\psi(x,y)=\psi_1(x,y)+\psi_2(x,y)$ be a generalized solution of equation (2.1). Then we call the function $u(x,y)$ a generalized solution of equation (4) in the neighborhood of the point z_0 if the latter is represented in this neighborhood by means of formula (5). The function $u(x,y)$ is called a solution of equation (4) in the domain G if it is a solution of this equation in the neighborhood of each point of the domain. We will consider only the classes of equations of form (4) or (3) which admit continuous generalized solutions inside the domain.

Since $A_1(x,y),A_2(x,y),A_3(x,y),f(x,y)\in L_p((\overline{G})$, $p>2$, the continuous solutions of equation (3) belong to the class $W_p^1(G)$ inside G. Therefore in this case the continuous solutions of the second order differential equation (4) will belong to the class in $W_p^2(G)$ [17].

Let us write a solution of problem (4) in the form $u(x,y)=v(x,y)+v^*(x,y)$, where v^* is a solution of the Dirichlet problem

$$\frac{\partial^2 v^*(x,y)}{\partial x^2}+\frac{\partial^2 v^*(x,y)}{\partial y^2}+A_1(x,y)\frac{\partial v^*(x,y)}{\partial x}+A_2(x,y)\frac{\partial v^*(x,y)}{\partial y}$$
$$+A_3(x,y)v^*(x,y)=f(x,y),\ (x,y)\in G, \tag{6}$$
$$v^*(x,y)=0,\ (x,y)\in\Gamma,$$

and v is a solution of the non-classical boundary value problem

$$\frac{\partial^2 v(x,y)}{\partial x^2}+\frac{\partial^2 v(x,y)}{\partial y^2}+A_1(x,y)\frac{\partial v(x,y)}{\partial x}+A_2(x,y)\frac{\partial v(x,y)}{\partial y}$$
$$+A_3(x,y)v(x,y)=0,\ (x,y)\in G\setminus\bigcup_{k=1}^{m}\gamma_k,$$
$$v(x,y)=0,\ (x,y)\in\Gamma, \tag{7}$$
$$\frac{\partial v(x_k^+,y)}{\partial x}-\frac{\partial v(x_k^-,y)}{\partial x}$$
$$=\sigma_k\frac{\partial v(1,y)}{\partial x}+\sigma_k\frac{\partial v^*(1,y)}{\partial x},\ 0\le y\le 1\ \ k=1,\dots,m.$$

As is known [18], problem (6) has a unique solution, which belongs to the space $W_2^2(G)\cap \overset{0}{W}{}_2^1(G)$. Hence it remains for us to investigate problem (7). To this end, we will investigate the iteration process

$$\frac{\partial^2 v^{n+1}(x,y)}{\partial x^2}+\frac{\partial^2 v^{n+1}(x,y)}{\partial y^2}$$
$$+A_1(x,y)\frac{\partial v^{n+1}(x,y)}{\partial x}+A_2(x,y)\frac{\partial v^{n+1}(x,y)}{\partial y}$$
$$+A_3(x,y)v^{n+1}(x,y)=0,\ (x,y)\in G\setminus\bigcup_{k=1}^{m}\gamma_k, \tag{8}$$
$$v^{n+1}(x,y)=0,\ (x,y)\in\Gamma,$$
$$\frac{\partial v^{n+1}(x_k^+,y)}{\partial x}-\frac{\partial v^{n+1}(x_k^-,y)}{\partial x}$$
$$=\sigma_k\frac{\partial v^n(1,y)}{\partial x}+\phi_k(y),\ 0\le y\le 1,\ k=1,\dots,m,\ n=0,1,2,\dots,$$

where $\phi_k(y)=\sigma_k\dfrac{\partial v^*(1,y)}{\partial x}$, $v^0(x,y)$ is the initial approximation which can be taken equal to zero.

Let $G(x,y,\xi,\eta)$ is Green's function, $\max\{\sigma_k\}<1/\sum_{k=1}^{m}\left\|\dfrac{\partial G(x,y,x_k,\eta)}{\partial x}\right\|$, then the iteration sequence (8) converges and, thereby, the existence and uniqueness of the solution of problem (4) in the space $W_2^2(G\setminus\bigcup_{k=1}^{m}\gamma_k)\cap \overset{0}{W}{}_2^1(G)$ is proved.

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