საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 12, №3, 2018
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 12, no.3, 2018

Mathematics

On the Estimation of the Difference of Boundary Values of Functions Quasiconformally Mapping Close Domains onto the Unit Circle

Lela Zivzivadze* and Erekle Japaridze*

**Faculty of Exact and Natural Sciences, Akaki Tsereteli State University, Kutaisi, Georgia*

(Presented by Academy Member Vakhtang Kokilashvili)

ABSTRACT. We consider the quasiconformal mappings of close domains onto the unit circle. Difference of the boundary values of quasiconformally mapping functions is estimated with respect to the order of closeness of domains. *© 2018 Bull. Georg. Natl. Acad. Sci.*

Key words: simply connected domain, Beltrami system of equations, principal homeomorphism of complex plane, quasiconformal mapping

On the complex plane **C** let us consider a simply connected domain $G,\quad 0\in G$, bounded by the smooth contour Γ . Assume that the equation of the boundary Γ is given in the parametric form

$$t=g(\tau),\quad 0\le\tau\le 2\pi,\quad g(0)=g(2\pi).$$

Let the function $f(t)$ of a point $t\in\Gamma$ be given on the curve Γ. Assuming $t=g(\tau)$, we can consider this function as a function of the parameter τ and again denote it by $f(\tau)$. If $f(\tau)$ satisfies the Hölder condition with exponent α , $0<\alpha<1$, then we will write $f\in C_\alpha[0,2\pi]$. If $f(\tau)$ and its derivative $f'(\tau)$ are continuous on a segment $0\le\tau\le 2\pi$ and if, in addition, $f'\in C_\alpha[0,2\pi]$, $0<\alpha<1$, then we write $f\in C'_\alpha[0,2\pi]$. If $g\in C'_\alpha[0,2\pi]$ we write $\Gamma\in C'_\alpha,\quad 0<\alpha<1$.

Norms of elements in spaces $C_\alpha[0,2\pi]$ and $C'_\alpha[0,2\pi]$ are defined as follows:

$$\|f\|_{C_\alpha}=\|f\|_C+H(f,\Gamma,\alpha),\qquad (f\in C_\alpha[0,2\pi]),$$
$$\|f\|_{C'_\alpha}=\|f\|_C+\|f'\|_C+H(f',\Gamma,\alpha),\qquad (f\in C'_\alpha[0,2\pi]),$$

where

$$\|f\|_C=\operatorname*{Sup}_{\tau\in[0,2\pi]}|f(\tau)|,\qquad H(f,\Gamma,\alpha)=\operatorname*{Sup}_{\substack{\tau_1,\tau_2\in[0,2\pi]\\ \tau_1\ne\tau_2}}\frac{|f(\tau_1)-f(\tau_2)|}{|\tau_1-\tau_2|^\alpha}.$$

Assume $\Gamma \in C'_\alpha$, $0<\alpha<1$. Let us consider another simply connected domain $\tilde{G}$ on the complex plane with the equation of the boundary $\tilde{\Gamma}$

$$\tilde{t}=\tilde{g}(\tau), \quad 0\le\tau\le 2\pi, \qquad \tilde{g}(0)=\tilde{g}(2\pi),$$

and $\tilde{\Gamma}\in C'_\alpha$.

Definition. The domain $\tilde{G}$ is called ε-close $(\varepsilon>0)$ to the domain G if the following conditions are fulfilled

$$|g(\tau)-\tilde{g}(\tau)|\le\varepsilon, \quad (\tau\in[0,2\pi]), \qquad \|g'-\tilde{g}'\|_{C_\alpha}\le\varepsilon. \tag{1}$$

Clearly, an infinite family of domains ε-close to G is formed for any $\varepsilon>0$. We denote such a set by $\Omega(G;\varepsilon)$.

Let G_0 be a simply connected domain of the complex domain whose boundary $\partial G_0=\Gamma_0\in C'_\alpha$ $(0<\alpha<1)$ is such that $G\subset G_0$ and for any $\tilde{G}\in\Omega(G;\varepsilon)$ $(0<\varepsilon<1)$, $\tilde{G}\subset G_0$.

In the complex plane we consider the Beltrami system of equations in the complex form

$$W_{\bar{z}}=q(z)W_z, \tag{2}$$

where it is assumed that $q\in L_p(\mathbf{C})$ $p>1$, $|q(z)|\le Q_0<1$.

Suppose the coefficient $q(z)$ of the equation (2) satisfies the condition $q\in C'_\gamma\left(\overline{G_0}\right)$ and $\tilde{W}(z)$ $(\tilde{W}(\infty)=\infty,\ z^{-1}\tilde{W}(z)\to 1$ as $z\to\infty)$ is the basic homeomorphism of the system (2) constructed by I. Vekua's scheme [1] with the coefficient $\tilde{q}(z)$

$$\tilde{q}(z)=\begin{cases} q(z) & \text{for } z\in\overline{G}_0, \\ 0 & \text{for } z\in C\setminus\overline{G}_0. \end{cases}$$

It is known that [2] under the above conditions $\tilde{W}_z$ and $\tilde{W}_{\bar{z}}$ satisfy the Hölder condition in the domain $\overline{G}_0$ with exponent γ_0 where $0<\gamma_0<\min\{\alpha;\gamma\}$.

According to [3], the function represented by the integral

$$\phi(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\varphi(t)\,d\tilde{W}(t)}{\tilde{W}(t)-\tilde{W}(z)}, \tag{3}$$

where $d\tilde{W}(t)=\tilde{W}_t(t)dt+\tilde{W}_{\bar{t}}(t)d\bar{t}$ and $\varphi(t)$ denotes a given Hölder-continuous function on Γ, is a solution of the system (2) in G, while in $\mathbf{C}\setminus\overline{G}$ it is a holomorphic function and also $\phi(\infty)=0$.

As is known [3], the function

$$f(z)=\left[\tilde{W}(z)-\tilde{W}(0)\right]\exp\left\{\frac{1}{\pi i}\int_\Gamma\frac{v(t)\,d\tilde{W}(t)}{\tilde{W}(t)-\tilde{W}(z)}+ic\right\},$$

satisfies the Beltrami equation (2) and conformally maps the domain G onto the unit circle with the fulfillment of the conditions: $f(0)=0$, $f(z_1)=1$ (z_1 is any point of the line Γ) if $v(t)$ is a solution of the integral equation

$$v(t)+\frac{1}{\pi}\int_\Gamma v(\tau)\,d\arg[\tilde{W}(\tau)-\tilde{W}(t)]=-\ln|\tilde{W}(t)-\tilde{W}(0)|, \tag{4}$$

where

$$c = -\arg[\tilde{W}(z_1) - \tilde{W}(0)] - \frac{1}{\pi}\int_{\Gamma} v(\tau)\, d\ln|\tilde{W}(\tau) - \tilde{W}(z_1)|.$$

In a similar way we construct the function

$$\tilde{f}(z) = [\tilde{W}(z) - \tilde{W}(0)]\exp\left\{\frac{1}{\pi i}\int_{\tilde{\Gamma}} \frac{\tilde{v}(\tau)\, d\tilde{W}(\tilde{t})}{\tilde{W}(\tilde{t}) - \tilde{W}(z)} + i\tilde{c}\right\},$$

which quasiconformally maps the domain $\tilde{G} \in \Omega(G;\varepsilon)$ onto the unit circle with the fulfillment of the conditions $\tilde{f}(0) = 0,\quad \tilde{f}(\tilde{z}_1) = 1$ ($\tilde{z}_1$ is any point of the line $\tilde{\Gamma}$), where the function $\tilde{v}(\tilde{t})$ of the Hölder class is a unique solution of an integral equation of type (4) derived for the domain $\tilde{G}$ and the line $\tilde{\Gamma} \in C'_\alpha$ where

$$\tilde{c} = -\arg[\tilde{W}(\tilde{z}_1) - \tilde{W}(0)] - \frac{1}{\pi}\int_{\tilde{\Gamma}} \tilde{v}(\tau)\, d\ln|\tilde{W}(\tau) - \tilde{W}(\tilde{z}_1)|.$$

Aim is to estimate the difference of the boundary values of f and $\tilde{f}$ by the nearness of the ε-close domains G and $\tilde{G}$. The following theorem is true.

Theorem. Let f and $\tilde{f}$ quasiconformally map the domains G and $\tilde{G} \in \Omega(G;\varepsilon)$ $(0 < \varepsilon < 1)$ onto the unit circle with the assumption that $f(z_1) = 1$ and $\tilde{f}(\tilde{z}_1) = 1$ where $z_1 = g(\tau_1)$, $\tau_1 \in [0, 2\pi]$ is any point on Γ for which the condition $\mathrm{Re}(\tilde{W}(z_1) - \tilde{W}(0)) > 0$ is fulfilled and $\tilde{z}_1 = \tilde{g}(\tau_1) \in \tilde{\Gamma}$. Then for any $\tau \in [0, 2\pi]$ and the corresponding boundary values $f(t) = f(g(\tau))$ and $\tilde{f}(\tilde{t}) = \tilde{f}(\tilde{g}(\tau))$ we have the estimate

$$|f(t) - \tilde{f}(\tilde{t})| \le M(G;\tilde{W}) \cdot \varepsilon^{\gamma_0/2},$$

where γ_0 is the aforementioned constant from the properties of the homeomorphism $\tilde{W}$ and the constant $M(G, \tilde{W})$ does not depend on ε and is completely defined by prescribing the initial domain G and the homeomorphism $\tilde{W}(z)$.

Note that for conformal mappings the similar theorem was established in [4].

მათემატიკა

მახლობელი არეების ერთეულოვან წრეზე კვაზიკონფორმულად ამსახველი ფუნქციების სასაზღვრო მნიშვნელობების სხვაობის შეფასების შესახებ

ლ. ზივზივაძე* და ე. ჯაფარიძე*

**აკაკი წერეთლის სახელმწიფო უნივერსიტეტი, ზუსტ და საბუნებისმეტყველო მეცნიერებათა ფაკულტეტი, ქუთაისი, საქართველო*

(წარმოდგენილია აკადემიის წევრის ვ.კოკილაშვილის მიერ)

ნაშრომში განიხილება მახლობელი არეების ერთეულოვან წრეზე კვაზიკონფორმულად ამსახველი ფუნქციები. მახლობელი არეების სიახლოვის მიხედვით შეფასებულია აღნიშნული არეების ერთეულოვან წრეზე კვაზიკონფორმულად გადამსახავი ფუნქციების სასაზღვრო მნიშვნელობების სხვაობა.

REFERENCES

1. Vekua I.N. (1988) Generalized analytic functions. M.
2. Manjavidze G.F. (1967) A boundary value problem of linear conjugation with displacement and its relation to the theory of generalized analytic functions. *Trans. Tbil. Math. Inst.*, **33:** 82-87.
3. Kveselava D.A., Samsonia Z.V. (1980) On the quasiconformal mapping of domains. *Metrich. Voprosi Teorii Funktsii* (*Metric Problems of Function Theory).* Naukova Dumka, Kiev, 53-65.
4. Zivzivadze L., Japaridze E. (2005) On estimation of exactness in conformal mappings of neighbouring domains. *Bull. Georg. Acad. Sci.* **171**, 3: 420-424.

Received June, 2018