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Physics

Quantization in the Topological Fields – Quantum Chiral Versus Skyrme Model

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ABSTRACT. Owing to the fact that both the Skyrme Lagrangian and the chiral non-linear sigma model lead to the same expression for static energy (mass) of soliton, it is natural to suggest that their solutions may be complimentary to each other. We propose that in many problems non-linear sigma model can replace the Skyrme model, as simplest one, which was demonstrated earlier by sufficient accuracy. In this paper the quantization of rotational and breathing modes are investigated. After quantization the corresponding profile functions do not satisfy needed boundary conditions, so they do not guarantee the stabilization of chiral soliton. Then it is shown that the inclusion of the pion mass term improves the situation – the Hamiltonian, obtained after quantization consists of effective potential, which has a lower bound. This fact gives hope to believe that in quantized chiral non-linear sigma model accounting the pion mass term should have soliton-like stable solutions. As regards of the numerical value of nucleon mass, derived by WKB solution, it is much lower than experimental one. Reasoning from the fact that the profile functions considered earlier in literature gave, as a rule, exceeding actual value, we can suppose that it is not excluded an existence of such profile functions that will improve the numerical correspondence as well. *© 2018 Bull. Georg. Natl. Acad. Sci.*

Key words*:* sigma model, chiral symmetry, Skyrme, numerical solutions

Recently [1] interactions between individual 3D skyrmions have been measured by physicists in China, Sweden, Russia and Germany. Their study shows that the magnetic quasiparticles feel both attractive and repulsive forces, depending on the strength of an applied magnetic field. As well as providing insights into the fundamental physics of magnetic materials, the research could lead to the development of devices that store data using skyrmions.

Skyrmions were first proposed as a new type of fundamental particle in the 1950s by British physicist T. Skyrme [2]. While these hypothetical particles have never been seen, certain collective particle-like excitations (quasiparticles) in magnetic solids have been shown to behave like skyrmions. Skyrmions can be extremely small and be manipulated using relatively small amounts of energy. Together, these properties suggest that skyrmions could be used to make dense and energy efficient computer memories.

The authors of this article also did theoretical calculations, which suggest that the observed interactions between skyrmions and skyrmion-edge are real – rather than the result of skyrmions being pinned by defects in the FeGe nanostripe.

In the light of this observation it is reasonable to review the theoretical foundation of skyrmions and compare with the simplest chiral solitons in particle physics.

It is well known that the Skyrme model, as a non-linear chiral theory of pions, provides an approximate description of hadronic physics in the low-energy limit [3, 4]. In this theory the nucleon emerges as a non-perturbative solution of the field equations, or more precisely as a topological soliton. This model is also seen as a prototype which might be applicable in various physical context where one could expect soliton solutions to occur (e.g. condensed matter physics - baby skyrmions, wrap branes,…), more recently, this model was applied for explanation for newly discovered hadronic states.

Let us remember that the original Skyrme Lagrangian is a naïve extension of the non-linear sigma model consisting of the fourth-order field derivative term. The Lagrangian of the non-linear sigma model has a form

$$L = -\frac{F_\pi^2}{4} Tr[\partial_\mu U^+ \partial^\mu U], \tag{1}$$

where $F_\pi = 93MeV$ is the pion decay constant, and U is an $SU(2)$ matrix, transforming as $U \to AUB^{-1}$ under chiral $SU(2)\times SU(2)$. We can look for static solutions using Skyrme's "hedgehog" anzatz

$$U = U_0 = \exp[i\tau \cdot nF(r)] \qquad n \equiv r / r. \tag{2}$$

The topological charge equals to

$$Q = \frac{i}{24\pi_2} \varepsilon^{ijk} \int_0^\infty d^3x Tr [U_0^+ \partial_i U_0][U_0^+ \partial_j U_0][U_0^+ \partial_k U_0] = [F(0) - F(\infty)] / \pi. \tag{3}$$

So, if the profile function $F(r)$ satisfies the boundary condition $F(0) = n\pi$ (n being integer) and $F(\infty) = 0$, then $Q = n.$ Mass of hedgehog configuration is given by

$$M_{cl} = 2\pi F_\pi^2 \int_0^\infty dr r^2 \left[\left(\frac{dF}{dr} \right)^2 + \frac{2}{r^2} \sin^2 F(r) \right]. \tag{4}$$

Corresponding Euler-Lagrange equation is the following:

$$r^2 \frac{d^2F}{dr^2} + 2r \frac{dF}{dr} = \sin(2F(r)). \tag{5}$$

The Euler-Lagrange equation results from the extreme condition for mass functional (4). Usually, when the equation cannot be solved analytically, one tries numerical methods or look for minimum of mass functional using trial profile functions (and paying no attention to equation of motion). Obviously satisfactory description of static properties of baryons is always possible by suitable choosing appropriate trial function.

Since the proper large –N effective theory is unknown, people choose the simplest Skyrme model [2]

$$L = \frac{1}{16} F_\pi^2 Tr\left(\partial^\mu U \partial_\mu U^+\right) + \frac{1}{32e^2} Tr\left[(\partial_\mu U)\mathrm{U}^+, (\partial_\nu U)\mathrm{U}^+\right]^2 \equiv L^{(2)} + L^{(4)}. \tag{6}$$

From this Lagrangian, using the Skyrme ansatz (2) we get the expression for soliton mass:

$$M_{Sk} = 4\pi \int_0^\infty r^2 dr \left\{ \frac{1}{8} F_\pi^2 \left[\left(\frac{\partial F}{\partial r} \right)^2 + 2 \frac{\sin^2 F}{r^2} \right] + \frac{1}{2e^2} \frac{\sin^2 F}{r^2} \left[\frac{\sin^2 F}{r^2} + 2 \left(\frac{\partial F}{\partial r} \right)^2 \right] \right\} \tag{7}$$

Introducing a dimensionless variable $\tilde{r} = eF_\pi r$ one can derive the variational equation from the above expression. It has the following form

$$\left(\frac{1}{4}\tilde{r}^2+2\sin^2 F\right)F''+\frac{1}{2}\tilde{r}F'+\sin 2FF'^2-\frac{1}{4}\sin 2F-\frac{\sin^2 F\sin 2F}{\tilde{r}^2}=0. \tag{8}$$

We see that there is no trace of the model parameters, so this equation may be solved numerically. It was carried out in [3] and has the form, shown in Figure.

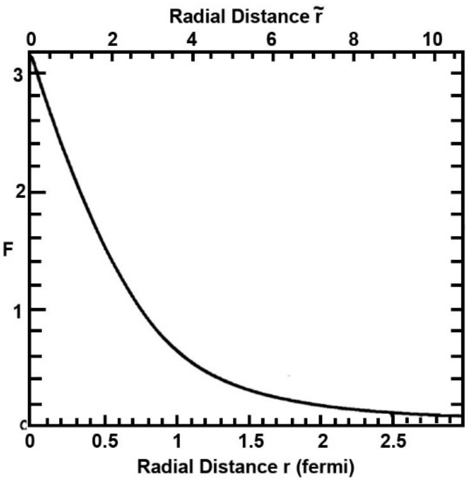

Figure. The numerical solution of eq. (7) (This figure is borrowed from the paper of G.S. Adkins, et al. [3]).

If we minimize the expression (7) for M_{Sk} with respect to chiral angle $F(r)$, evidently, we obtain the equation of motion. Therefore for calculation of minimum of mass, we can use here the equation of motion, which after using the virial theorem means:

$$E^{(2)}=E^{(4)}. \tag{9}$$

Therefore

$$M_{sk}=2\pi F_\pi^2\int_0^\infty r^2 dr\left[F'^2+\frac{2\sin^2 F}{r^2}\right]. \tag{10}$$

It is very strange that this expression coincides with the mass of sigma model (4).

Because the stability of topological configurations in the classical sigma model is not achieved contrary to Skyrme model, we have resort to quantized version of chiral Lagrangian. For quantization of the breathing mode together with rotational ones let us substitute ansatz $U=A(t)U_0\left(\frac{r}{R(t)}\right)A^{-1}(t)$ into non-linear sigma model Lagrangian. Denoting $X=R^{2/3}$ (note that both R and X are dimensionless), we obtain

$$L=\frac{a}{F_\pi}\left(\frac{dX}{dt}\right)^2-bF_\pi X^{2/3}-\frac{X^2}{F_\pi}ITr\left(\dot{A}\dot{A}^+\right). \tag{11}$$

From which we can construct corresponding Hamiltonian and using variational principle derive the equation for profile function. It has the form

$$r^2\frac{d^2F}{dr^2}\left(Nr^2-1\right)+2r\frac{dF}{dr}\left(2Nr^2-1\right)+\sin\left(2F(r)\right)=0, \tag{12}$$

where $N=4b/27a$. This equation is a scale invariant, but has no self-consistent solutions for $N\neq 0$, obeying to needed boundary conditions mentioned after the Eq. (3). So the quantization of breathing mode in chiral invariant non-linear model cannot lead to stable soliton solutions.

Look now how can standard pion mass term affect the spectrum of Schrödinger equation. It gives

$$\left[-\frac{d^2}{dZ^2}+\left(\frac{b^3}{4a}\right)^{1/3}Z^{2/3}+\left(3/4+2T(T+1)\frac{a}{I}\frac{1}{Z^2}+\frac{m_\pi^2}{F_\pi^2}\frac{C}{4}\frac{Z^2}{a}\right)\right]\Phi=\frac{E}{F_\pi}\Phi. \tag{13}$$

Parameters entering (10) were exhibited in [4]. It follows that the effective potential in this equation has the form

$$W_{eff}(Z)=\alpha Z^{2/3}+\beta/Z^2+\gamma Z^2, \qquad \gamma\equiv m_\pi c/4aF_\pi^2. \tag{14}$$

It is clear that this potential will have a non-trivial minimum if at least one of the two parameters - α or γ is nonzero. So there is a hope that the new term ensures stability.

Introducing a new variable $Y=\gamma^{1/4}Z$ we transform this equation into

$$\left[-\frac{d^2}{dY^2}+\alpha\gamma^{-2/3}Y^{2/3}+\beta/Y^2+Y^2\right]\Phi(Y)=\left(\frac{E}{\gamma^{1/2}F_\pi}\right)\Phi(Y). \tag{15}$$

Let us look at WKB solution

$$E_n=m_\pi(c/4a)^{1/2}\left[W_{eff}(Y_0)+\sqrt{2W''_{eff}(Y_0)}(n+1/2)\right]. \tag{16}$$

Here Y_0 must be derived from Cardano's formula, as solution of cubic equation

$$\beta/Y_0^2=\alpha\gamma^{-2/3}Y_0^{2/3}/3+Y_0^2. \tag{17}$$

By suitable choice of parameters, the following result follows for WKB energy [4]

$$E_n=(m_\pi/2)\left[4n+2+\sqrt{4+6T(T+1)}\right]. \tag{18}$$

We see that the quantization of rotational modes and incorporation of pion mass term leads to soliton solutions with ground state energy bounded from below. But the profile functions in that case turn out to be pathological. Stability of the soliton can be achieved by quantizing vibrational mode with massive pions. The only question that remains is whether the profile function minimizing energy is smooth or pathological.

The other side of the problem is that the numerical value of soliton mass, obtained by (15) is much less than the nucleon mass. So it will not suit for description of static properties of baryons. But the fact of existence of energy spectrum bounded from below is important. Besides the profile functions usually used give an estimate of nucleon mass exceeding actual value. So as long as the model with chiral symmetry breaking produce much lower values for soliton mass it gives better opportunities to find profile functions corresponding to experimental value of nucleon mass.

This possibility was verified in [4], While stability problem remains still open, but the numerical coincidence within the 30% strengthened the belief that the better profile functions can be found. Therefore we may think that the non-linear sigma model can have analogous soliton solutions and the close predictions for nucleon static properties.

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