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*Mathematics*

# On the Weak Distribution of Signed Measure in the Hilbert Space

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**ABSTRACT. In the paper, the properties of weak distributions of the signed measure in the infinite-dimensional Hilbert space are considered. Necessary and sufficient properties for inducing weak distributions by measure are found.** *© 2018 Bull. Georg. Natl. Acad. Sci.*

**Key words:** measurable space, cylinder sets, $\delta$-algebra

Consider real, separable Hilbert space $H$ and $\delta$ -algebra $\mathcal{B}$ of its Borel sets. Thus, the measurable space $(H, \mathcal{B})$ in sense of [1] is considered. Denote by $\mathcal{P}$ the set of all finite-dimensional orthogonal projections on $H$ and by $\mathcal{N}$ the set of all finite-dimensional subspaces of Hilbert space $H$ . Therefore, for any $P \in \mathcal{P}$ we have $PH \in \mathcal{N}$. The set $P_L^{-1}(A)$ is the cylindrical set in $H$ , where $P_L \in \mathcal{P}$ is the projector on $L \in \mathcal{N}$ and $A$ the so-called base of cylinder is the Borel set in $L$. Cylinder sets in $H$ with the bases in $L$ generate $\sigma$-algebra, which is denoted by $\mathcal{B}$ $(L)$. It is clear that $\mathcal{B}$ $(L) \in \mathcal{B}$. Union of all $\sigma$-algebras $\mathcal{B}$ $(L)$ is an algebra, which is denoted by $\mathcal{B}_0$. It is called the algebra of cylinder sets. It is known that the $\sigma$-closure of $\mathcal{B}_0$ is $\mathcal{B}$.

For any finite-dimensional subspace $L$ of the Hilbert space $H$, consider signed finite measure $\mu_L$ defined on the Borel $\sigma$-algebra $\mathcal{B}_L$ of the sets from $L$ . Assume that the family of measures $\{ \mu_L, L \in \mathcal{N}\}$ is adapted in the sense that for all spaces $L_1 \subset L_2, L_i \in \mathcal{N}$, $i$ =1,2, and for any Borel set $A \in \mathcal{B}_L$, we have

$$\mu_{L_1}(A) = \mu L_2(P_{L_1}^{-1}(A) \cap L_2)\,, \tag{1}$$

The family of measures $\{ \mu_L, L \in \mathcal{N}\}$ satisfying the condition (1) and defined for any finite dimensional spaces $L$ for a given separable Hilbert space $H$ is called a weak distribution. We use for this weak distribution the designation $\mu^* = \{ \mu_L, L \in \mathcal{N}\}$. If any measure $\mu_L, L \in \mathcal{N}$ is non-negative, then the weak distribution $\{ \mu_L, L \in \mathcal{N}\}$ is called positive and we write $\mu^* \geq 0$. In case, when the chain of growing finite dimensional subspaces $\{L_n\}$, $L_n \subset L_{n+1}$, $L_n \in \mathcal{N}$, $n \in N$ is considered, for which $\bigcup_{n=1}^{\infty} L_n$ is tight in $H$, then the sequence of measures $\{\mu_{Ln}$, L $\in \mathcal{N}\}$ on $\mathcal{B}_{Ln}$ and satisfying the condition

$$\mu_{Ln}(A) = \mu_{Ln+1}(P_{Ln}^{-1}(A) \cap L_{n+1})$$

is called the sequence of finite dimensional distributions (mainly so-called sequence of non-negative measures).

In this paper, we use the terminology of a weak distribution for both cases. The full variation $|\mu^*|$ of the weak distribution $\mu^* = \{\mu_L, L \in \mathcal{N}\}$ is called the weak distribution (sequence of finite dimensional distributions) $|\mu^*| = \{|\mu_L|, L \in \mathcal{N}\}$ where $|\mu_L|$ denotes the full variation of measures $\mu_L = \mu_L^+ - \mu_L^-$. Certainly, $|\mu_L| = \mu_L^+ + \mu_L^-$.

If we know a priori that there exists the measure $\mu$ on $H$, then by related given weak distribution $\{\mu_{L_n}$ $L \in \mathcal{N}\}$ it is possible uniquely to reconstruct the measure. Indeed, knowing $\mu_{L_n}$ for the sequence of subspaces $\{L_n\}$, such that $L_n \subset L_{n+1}$ and $\bigcup_{n=1}^{\infty} L_n$ is tight in $H$, first define by $\mu_{L_n}$ the measure on $\mathcal{B}(L_n)$, so that $\mu$ is defined on the algebra $\mathcal{B}_0 = \bigcup_{n=1}^{\infty} L_n$. Since this algebra generates $\mathcal{B}$, then it is clear that $\mu$ is uniquely defined on $\mathcal{B}$ too.

However, in practice, for a given weak distribution, it is not always known whether or not this distributions generated by some sort of measure on a Hilbert space. To establish this fact additional conditions are needed. Different approaches to these questions and the results can be found in a vast literature [2-3]. Denote by $C_L(H)$ the space of continuous and finite functions on $L$. $C_L(H)$ is a Banach space in uniform norm $\|\varphi\|_{C_L(H)} = \sup_{x \in L} |\varphi(x)|$. Further, let $C^*(H)$ be the space of continuous and finite cylindrical functions. Recall, that a function $\varphi(x)$ is called cylindrical if there is a $\mathcal{B}_L$ measurable function $\varphi_L$, such that the following representation is true

$$\varphi(x) = \varphi_L(P_L x), \tag{2}$$

In this case, $L$ is called the support of this function. In the definition of $C^*(H)$ in addition we request that $\varphi_L \in C_L(H)$ for some $L$. $C^*(H)$ is a linear normed space. In the uniform norm, its closure coincides with $C(H)$ - the space of continuous and finite functions on $H$, with the norm $\|\varphi\|_{C_L(H)} = \sup_{x \in L} |\varphi(x)|$. Denote by $C^L(H)$ the space of continuous and finite cylindrical functions with base in $L$. Then $C^*(H) = \bigcup_L C^L(H)$.

For the functions from space $C^*(H)$, an integral on weak distribution $\{\mu_L, L \in \mathcal{N}\}$ can be defined. This integral following [1] is written in the form

$$\int_H \varphi(x)\mu^*(dx) = \int_L \varphi_L(x)\mu_L(dx), \tag{3}$$

where $\varphi_L$ is the function from (2). The measures $\mu_L$ are adapted, so the definition (3) is correct.

The integral, defined in such a way, satisfies the properties of the linear operation

$$\int_H (c_1\varphi_1(x) + c_2\varphi_2(x))\mu^*(dx) = c_1 \int_H \varphi_1(x)\mu^*(dx) + c_2 \int_H \varphi_2(x)\mu^*(dx),$$

where $c_i \in R, \varphi_i \in C^*(H), i = 1.2$.

From the above definitions it is easy to see that the integral in the weak distribution can also be defined not only for continuous, but for cylindrical bounded measurable functions, or direct determination from the beginning, or as the limit of continuous cylindrical functions. But it turns out that this is not all. The integral can be extended further to some non-cylindrical functions.

In the general case, i.e. for sign-changing weak distributions, such a criterion is known

**Theorem 1 ([3]).** *Let $\mu^*$ be the weak distribution such that for any $\varepsilon > 0$ and for some $\rho > 0$, there exists an environment $U$ f zero in $(H, \tau s(H))$, such that*

$$\left|\mu^*(A \cap \{y \in H : |(x,y)| > \rho\})\right| < \varepsilon \tag{4}$$

*for any $A \in \mathcal{B}_0$ and $x \in U$ Then $\mu^*$ is generated by measure from $H$ and conversely. And if the initial weak distribution is positive, then for the validity of the statement it is necessary and sufficient to find for any $\varepsilon > 0$ a compact $K$ such that $\mu_L(L - P_L(K)) \le \varepsilon$ for any $L \in \mathcal{N}$.*

Consider a bounded measurable function $f(x) = f(x_1, x_2, ... x_n)$ in $R^n$. Then, for any $y_i \in H, i = 1.2,...,$ $n$ we can consider the cylindrical function

$$f(x) = f((x, y_1), (x, y_2), ..., (x, y_n)) \tag{5}$$

and the integral makes sense

$$\int_H f(x)\mu^*(dx) = \int_{Ln} f(t_1, t_2, ..., t_n), \mu_{y_1 y_2,...,y_n}(dt_1, dt_2, ..., dt_n),$$

where $L_n$ is the finite dimensional subspace of $H$ and unitarily isomorphic to $R^n$, $\mu_{y_1, y_2,...,y_n}$ image of the weak distribution $\mu^*$ in $L_n$. If $f(x)$ is the uniform limit of cylindrical functions $f_n(x)$, then we assume that

$$\int_H f(x)\mu^*(dx) = \lim_{n\to\infty} \int_H f_n(x)\mu^*(dx)$$

when this limit exists.

We can specify the classes of non-cylindrical functions for which the integral $\int_H f(x)\mu^*(dx)$ can be extended. We describe one of these classes. Let $f(x)$ be continuous on $(-\infty, +\infty)$, real, bounded by limited unit, positive, decreasing and vanishing at infinity function $\lim_{n\to\infty} f(x) = 0.$ The set of such functions is denoted by $\mathcal{F}$. We show that for any function from $\mathcal{F}$ there exists the integral $\int_H f(-\|x\|)\mu^*(dx).$ Indeed, the function $f_n(x) = f(-P_n(x), P_n(x))$ is cylindrical and uniformly converges to $f(x).$ Here $P_n$ is the orthogonal projection on the $n$-dimensional subspace $L_n \subset H.$ At the same time, $L_n$ are chosen so that they increase and $\bigcup_{n=1}^{\infty} L_n$ is tight in $H.$ For the numerical sequence $\alpha_n = \int_{Ln} f_n(x)\mu^*(dx), n = 1.2,...,$ When $n > m$ then

$$|\alpha_n - \alpha_m| = \left|\int_{Ln} f_n(x)\mu_n(dx) - \int_{Lm} f_m(x)\mu_m(dx)\right|$$

$$= \left|\int_{Ln} f_n(x)\mu_n(dx) - \int_{Ln} f_m(x)\mu_n(dx)\right| \le$$

$$\int_{Ln} |f_n(x) - f_m(x)||\mu_n|(dx) \le |\mu_n|(L_n)\sup_x |f_n(x) - f_m(x)|.$$

Because of the uniform convergence, the last expression can be made arbitrarily small at $m, n, \to \infty.$ Therefore, the number sequence $\{\alpha_n\}$ converges to some $\alpha$.

Denote by $\mathcal{W}(\mu^*)$ the set of integrable by $\mu^*$ functions. It is clear that $\mathcal{F} \subset \mathcal{W}(\mu^*)$. Since the total mass is the same for every $L$, then we will assume that $|\mu_L(L)| = 1$ and write $|\mu^*(H)| = 1.$

**Theorem 2.** To induce by measure the weak distribution $\mu^*$ it is necessary and sufficient that

$$\lim_{\varepsilon \downarrow 0} \int_H f(-\varepsilon\|x\|^2)\mu^*(dx) = f(0), f \in \mathcal{F} \tag{6}$$

**Proof.** If $\mu^*$ is induced by measure $\mu$, then using Fatou-Lebesque theorem

$$\lim_{\varepsilon\downarrow 0}\int_H f(-\varepsilon\|x\|^2)\mu^*(dx)=\lim_{\varepsilon\downarrow 0}\int_H f(-\varepsilon\|x\|^2)\mu(dx)=\int_H \lim_{\varepsilon\downarrow 0} f(-\varepsilon\|x\|^2)\mu(dx)=f(0)\mu(H)=f(0).$$

Conversely, suppose we have (6) and check the validity of (4). For any set $A\in\mathcal{B}_0$ there exists $L\in\mathcal{N}$, such that $A\in\mathcal{B}_L$ note. Note that since

$$\left|\mu^*(A\cap\{y\in H:|(x,y)|>\rho\})\right|\le\left|\mu^*(L\cap\{y\in H:|(x,y)|>\rho\})\right|,$$

then (4) it is sufficient to check for $A = L$.

Introduce the notation $A_x=\{y\in H:|(x,y)|>\rho\}$. By the condition (6) for any $0<\delta<1$, there is $\varepsilon_0>0$, such that

$$\int_H f(-\bar{\varepsilon}\|x\|^2)\mu^*(dx)>1-\delta, \text{ when } 0<\bar{\varepsilon}<\varepsilon_0.$$

So we write

$$1-\delta<\int_L f(-\bar{\varepsilon}\|x\|^2)\mu_L(dx)\le\int_{L\cap A_x} f(-\bar{\varepsilon}\|y\|^2)|\mu_L(dy)|+$$

$$\int_{L-L\cap A_x} f(-\bar{\varepsilon}\|y\|^2)|\mu_L(dy)|\le|\mu_L(L\cap A_x)|+\sup_{y\in L:|(x,y)|\le\rho}$$

$$f(-\bar{\varepsilon}\|y\|^2)(1-|\mu_L(L\cap A_x)|=\sup_{y\in L:|(x,y)|\le\rho} f(-\bar{\varepsilon}\|y\|^2)+(1-\sup_{y\in L:|(x,y)|\le\rho} f(-\bar{\varepsilon}\|y\|^2)|\mu_L(L\cap A_x)|.$$

From that we obtain

$$|\mu_L(L\cap A_x)|<1-\frac{\delta}{\sup_{y\in L:|(x,y)|\le\rho} f(-\bar{\varepsilon}\|y\|^2)}. \tag{7}$$

For an arbitrary $\rho>0$, consider the open sphere: $U=\{x\in H:\|x\|<\rho^2\}$. Using the inequality $|(x,y)|^2\le\|x\|\|y\|<\rho^2\|y\|$, we obtain the following estimation

$$\min_{x\in U}\sup_{y\in L:|(x,y)|\le\rho} f(-\bar{\varepsilon}\|y\|^2)\ge f(0)$$

And from (7)

$$|\mu_L(L\cap A_x)|<1-\frac{\delta}{1-f(0)}. \tag{8}$$

Fix first $\rho>0$, then for the given $\varepsilon>0$, choose $\delta$, such that $f(0)>\dfrac{\delta}{1-f(0)}$. $\delta$ determines $\varepsilon_0$. Finally, from (8) we obtain $|\mu_L(L\cap A_x)|<\varepsilon$ for any $x\in U$.

**Remark.** This theorem is a generalization of the Skorokhod result (see [1], Lemma 2) in the case of sign changing weak distribution. For positive measures in [1] the function $f(x)=e^x$ is used on $[0,\infty)$.

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# ნიშნიანი ზომის სუსტი განაწილების შესახებ ჰილბერტის სივრცეში

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