საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 12, №4, 2018
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 12, no.4, 2018

Mathematics

On the Structure of the Space of Generalized Analytic Functions

Grigori Giorgadze

Ivane Javakhishvili Tbilisi State University,
I.Vekua Institute of Applied Mathematics, Tbilisi, Georgia

(Presented by Academy Member Revaz Gamkrelidze)

ABSTRACT. **We discuss on algebraic structure of solutions space of regular Carleman-Bers-Vekua equation. Using higher derivation in sense of Bers of generalized analytic functions we construct sequence of real vector spaces and such way we classify solutions space. Periodicity properties of first kind pseudo-analytic functions we extend on second kind pseudo-analytic functions and proved similar results for solutions of Beltrami equations. Consequently we obtain periodicity of complex structures on Riemann surfaces. We also give several related problems and conjectures are also discussed.** *© 2018 Bull. Georg. Natl. Acad. Sci.*

Key words: pseudo-analytic function, minimum period, Bertrami equation, Carleman-Bers-Vekua equation, solution space

The complex structures of vector bundle (see [1]) is determined by the generalized Cauchy-Riemann operator with the kernel of either generalized analytic [2] or pseudo-analytic functions [3].

After the appearance of the monographs of I.Vekua [2] and L.Bers [3], generalized analytic functions, due to Vekua, and pseudo-analytic functions, due to Bers, became the subject of study for many mathematicians. In the frame of Vekua-Bers theory, the geometric (topological) properties of the solutions of elliptic systems on the plane became clear, establishing, in particular, the natural relation between the pseudo analytic functions and the quasiconformal mappings (here we mean the Carleman-Bers-Vekua and Beltrami equations, which always co-exist [1]). The equivalence between these two different approaches results in the theory of sufficiently wide function space. The main theorems of the theory of analytic functions can be extended to this space. Bers' and Vekua's methods result an identical entity in the cases of the Bers normalized generating pair and the Carleman-Bers-Vekua regular equations. In particular, the normalized generating pair induces the regular equation and vice versa. Hence, these approaches are naturally completing each other, producing functional spaces with richer properties. In other cases, when the equation is non-regular [1] or a Bers generating pair is not normalized, the corresponding space of the generalized analytic functions is studied less intensively. By all means, the theory is less organized than the Bers-Vekua theory.

In this paper we consider relationship between first and second kind pseudo-analytic functions in regular case the periodisity point of view.

Let $F(z), G(z)$ be two complex valued function in domain U of the complex plane $\mathbb{C}$, possessing Hölder continuous partial derivatives with respect to the real variables $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. Any pair such functions $(F(z), G(z))$ satisfying condition $\operatorname{Im}(\overline{F}G) > 0$ in U called *generating pair*. It is clear that for every point $z_0 \in U$ there exist unique pair $\lambda_0, \mu_0 \in \mathbb{R}$ of real numbers such that functions, defined in some domain such that $\operatorname{Im}(\overline{F}G) > 0$. A function $w = \varphi F + \psi G$, where φ and ψ are real, is called (F,G) -pseudo-analytic (or *pseudo-analytic of first kind*), if $\varphi_{\overline{z}} F + \psi_{\overline{z}} G = 0$. The function $\dot{w} = \varphi_z F + \psi_z G$ is called the (F,G) *-derivatives* of w and $\omega = \varphi + i\psi$ is called *pseudo-analytic function of second kind.* Every generating pair (F,G) has a successor (F_1, G_1) such that (F,G) derivatives are (F_1, G_1) pseudo-analytic. The successor is not uniquely determined. A generating pair (F,G) is said to have *minimum period n* if there exists generating pairs (F_i, G_i) such that (F_{i+1}, G_{i+1}) is a successor of (F_i, G_i) and $(F_n, G_n) = (F_0, G_0)$. If such *n* does not exist, (F,G) is said to have minimum period ∞.

It is known, that *w* is pseudo-analytic if *w* satisfies the following Carleman-Bers-Vekua equation

$$w_{\overline{z}} = aw + b\overline{w}, \tag{1}$$

where the function $a(z, \overline{z})$, $b(z, \overline{z})$ expressed by the generating pair (F,G) through the following identity

$$a = \frac{\overline{G}F_{\overline{z}} - \overline{F}G_{\overline{z}}}{F\overline{G} - \overline{F}G}, b = \frac{FG_{\overline{z}} - GF_{\overline{z}}}{F\overline{G} - \overline{F}G}. \tag{2}$$

Define also the the quantities

$$A = \frac{\overline{G}F_z - \overline{F}G_z}{F\overline{G} - \overline{F}G}, B = \frac{FG_z - GF_z}{F\overline{G} - \overline{F}G}. \tag{3}$$

The (F,G)-derivative $\dot{w}$ satisfies the following Carleman-Bers-Vekua equation

$$\dot{w}_{\overline{z}} = a\dot{w} - B\overline{\dot{w}}. \tag{4}$$

The functions *a, b, A, B* are called the *characteristic coefficients* of the generating pair (F,G) (see [4]).

It is known that for given functions *a, b, A,B* are characteristic coefficients of the generating pair if and only if they satisfy the system of differential equations

$$A_{\overline{z}} = a_z + b\overline{b} - B\overline{B}, B_{\overline{z}} = b_z + (\overline{a} - A)b + (a - \overline{A})B.$$

Denote by $\Omega(a,b)$ the solution space of the equation (1). The description of $\Omega(a,b)$ by period is given in following proposition:

Proposition 1 [5] *1) The space* $\Omega(a,b)$ *have period one if there exist a functions* A_0 *satisfying the equation*

$$A_{0\overline{z}} = a_z,\ A_0 - \tilde{A}_0 = \overline{a} - a + \frac{1}{b}(b_z + b_{\overline{z}}).$$

2) The space $\Omega(a,b)$ have period one if there exist functions A_0, A_1, B_0 satisfying the equation

$$A_{0\overline{z}} = a_z - b\overline{b} + B_0\overline{B}_0,\ B_{0\overline{z}} = b_z + (\overline{a} - A_0)b + (a - \overline{A}_0)B_0,$$

$$A_{1\overline{z}} = a_z + b\overline{b} - B_0\overline{B}_0,\ B_{0\overline{z}} = b_{\overline{z}} + (A_1 - \overline{a})B_0 + (\overline{A}_1 - a)b.$$

Proposition 2 *Let* (F,G) *be a generating pair of (1), then the generating pair of the adjoint equation*

$$w_{\overline{z}} = -aw - \overline{B}\overline{w}$$

is

$$F^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}, \quad G^* = \frac{2\overline{F}}{F\overline{G} - \overline{F}G}. \tag{5}$$

We prove that the characteristic coefficients, induced from adjoint generating pair (F^*, G^*), are equal to $-a$ and $-\overline{B}$. Indeed,

$$\frac{\overline{G}^* F^*_{\overline{z}} - \overline{F}^* G^*_{\overline{z}}}{F^*\overline{G}^* - \overline{F}^* G^*} = \frac{\frac{2F}{D}\left(\frac{2\overline{G}}{D}\right)_{\overline{z}} - \frac{2G}{D}\left(\frac{2\overline{F}}{D}\right)_{\overline{z}}}{\frac{2\overline{G}}{D}\frac{2F}{\overline{D}} - \frac{2G}{\overline{D}}\frac{2\overline{F}}{D}} = \frac{\frac{4F}{D}\left(\frac{\overline{G}}{D} - \frac{\overline{G}}{D^2}D_{\overline{z}}\right) - \frac{4G}{D}\left(\frac{\overline{F}_{\overline{z}}}{D} - \frac{\overline{F}}{D^2}D_{\overline{z}}\right)}{\frac{4}{D\overline{D}}(F\overline{G} - \overline{F}G)} =$$

$$= \frac{F\overline{G}_{\overline{z}} - G\overline{F}_{\overline{z}}}{D} - \frac{F\overline{G} - G\overline{F}}{D^2}D_{\overline{z}},$$

where $D = F\overline{G} - \overline{F}G$, $D = -\overline{D}$, $D_{\overline{z}} = F_{\overline{z}}\overline{G} + F\overline{G}_{\overline{z}} - \overline{F}_{\overline{z}}G - \overline{F}G_{\overline{z}}$. From (5) we have

$$a_1 = \frac{-F_{\overline{z}}\overline{G} - -F\overline{G}_{\overline{z}} - \overline{F}_{\overline{z}}G - \overline{F}G_{\overline{z}} + \overline{F}_{\overline{z}}G + \overline{F}G_{\overline{z}}}{D} = -\frac{F_{\overline{z}}\overline{G} - \overline{F}G_{\overline{z}}}{D},$$

from this follows, that $a = -a_1$.

Analogously as above

$$b_1 = \frac{F^* G^*_{\overline{z}} - G^* F^*_{\overline{z}}}{F^*\overline{G}^* - \overline{F}^* G^*} = \frac{\frac{G}{D}\overline{G}\left(\frac{\overline{F}}{D}\right)_{\overline{z}} - \frac{G}{D}\overline{F}\left(\frac{2\overline{G}}{D}\right)_{\overline{z}}}{\frac{G}{D}} = \frac{\overline{D}}{D}\left(\frac{\overline{G}\overline{F}_{\overline{z}}}{D} - \frac{\overline{G}\overline{F}}{D^2}D_{\overline{z}} - \frac{\overline{F}\overline{G}_{\overline{z}}}{D} + \frac{\overline{F}\overline{G}}{D^2}D_{\overline{z}}\right) =$$

$$= \frac{\overline{G}\overline{F}_{\overline{z}} - \overline{F}\overline{G}_{\overline{z}}}{D},$$

therefore,

$$\overline{b}_1 = -\frac{GF_z - FG_z}{D}$$

and hence $\overline{b}_1 = -\overline{B}$.

By definition [3] the pseudo-analytic functions corresponding to (5) satisfy the following holomorphic disc equation

$$w_{\overline{z}} = q(z)\overline{w}_z, \tag{6}$$

where

$$q(z) = \frac{F + iG}{F - iG}.$$

Proposition 3 *Holomorphic disc equation, corresponding to (4) is*

$$w_{\overline{z}} = -\overline{q(z)}\overline{w}_z.$$

Indeed, coefficient of holomorphic disc equation, corresponding to (4) expressed by the generating pair (F^*, G^*) of (5) as

$$q_1(z) = \frac{F^* + iG^*}{F^* - iG^*} = \frac{\frac{2\overline{G}}{D} + i\frac{2\overline{F}}{D}}{\frac{2\overline{G}}{D} - i\frac{2\overline{F}}{D}} = \frac{\overline{G} + i\overline{F}}{\overline{G} - i\overline{F}},$$

it means that $q_1 = -\overline{q}$.

Proposition 4 *If equation (1) has the period one, then equation (4) also has period one.*

The proof immediately follows from the proof of the preceding proposition.

Proposition 5 *The generating pair of the space* $\Omega(a,0)$ *is* (f, if)*, where* $f \neq 0$ *(and is solution of the equation* $f_{\overline{z}} = -af$ *).*

Indeed, $\mathrm{Im}(\overline{f}if) = |f|^2$, in other hand $F\overline{G} - \overline{F}G = f(-i\overline{f}) - \overline{f}(if) = -2i|f|^2$. From this

$$a_{(f,if)} = \frac{\overline{f}if_{\overline{z}} - f_{\overline{z}}(-i\overline{f})}{-2i|f|^2} = -\frac{f_{\overline{z}}}{f}$$

and

$$b_{(f,if)} = \frac{-fif_{\overline{z}} - f_{\overline{z}}if}{-2i|f|^2} = 0.$$

Consider the particular cases of this proposition. When f is constant, or is complex analytic, we obtain the space of holomorphic functions $\Omega(0,0)$.

Proposition 6 *If* f *is real and* $f \neq 0$*, then* $(f, \frac{i}{f})$ *generates the space* $\Omega(0, b)$.

The proof is obtained from direct calculation: $\mathrm{Im}\left(f\frac{i}{f}\right) = 1 > 0$ because $f = \overline{f}$;

$$a_{(f,\frac{i}{f})} = -\frac{f\left(\frac{if_{\overline{z}}}{f^2}\right) - f_{\overline{z}}\frac{i}{f}}{-2i} = 0, \quad b_{(f,\frac{i}{f})} = -\frac{-f\left(\frac{if_{\overline{z}}}{f^2}\right) - f_{\overline{z}}\frac{i}{f}}{-2i} = \frac{f_{\overline{z}}}{f}.$$

Proposition 7 *From* $w \in \Omega(a,0)$ *it follows, that* $\dot{w} \in \Omega(a,0)$.

By proposition 5 the generating pair of the space $\Omega(a, 0)$ is (f, if). The function $\dot{w}$ satisfies the equation (4), therefore it is necessary to calculate B from (3). It is easy, that $B = 0$.

In case, when the functions F, G are complex analytic, then from (2) it follows that we obtain the space of holomorphic functions $\Omega(0, 0)$, but this space is not "isomorphic" to the space of holomorphic functions generated by the pair $(1, i)$, because it follows from (3), B is not equal to zero. From this follows that this space has period $N > 1$ and that is shown in [5], the period of this space is equal to 2.

Remark 1. 1) There exists real analytic function b in the a neighborhood of the origin, such that the space $\Omega(0, b)$ has minimum period infinity.

2) For each positive integer N there exists a real analytic function b in the neighborhood of the origin, such that the space $\Omega(0, b)$ has minimum period N.

Remark 2. L. Bers obtained the necessary and sufficient condition that $\Omega(a, b)$ generated by (F, G) has the period one and proved that this condition is identity $\frac{F}{G} = \tau(y)$. L. Bers also proved, that if $\frac{F}{G} = \sigma(x)$, then the minimum period is at most two.

From above it follows that there always exists the space of pseudo-analytic functions, containing a given admissible function; moreover it is possible to construct the spaces $\Omega(a_1, b_1)$ and $\Omega(a_2, b_2)$ with nonempty intersection. Indeed, if (F_1, G_1) is the generating pair with characteristic coefficients A_1 and B_1 and (F_2, G_2) is other generating pair not equivalent to (F_1, G_1) with characteristic coefficients (F_2, G_2) and $F_1 = F_2$, then F_1 is a common element of the spaces $\Omega(a_1, b_1)$ and $\Omega(a_2, b_2)$.

If f be a real value positive function, then $(f, \frac{i}{f}))$ -pseodoanalytic functions of first kind are *p*-analytic functions with $p = f^2$ and corresponding pseudo-analytic function of second kind satisfies the Beltrami equation with coefficient $\frac{f^2+1}{f^2-1}$.

This work is supported by Shota Rustaveli National Science Foundation grant N FR17-96 .

მათემატიკა

განზოგადებულ ანალიზურ ფუნქციათა სივრცის სტრუქტურის შესახებ

გ. გიორგაძე

ივანე ჯავახიშვილის სახელობის თბილისის სახელმწიფო უნივერსიტეტი,
ილია ვეკუას სახელობის გამოყენებითი მათემატიკის ინსტიტუტი, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის რევაზ გამყრელიძის მიერ)

განხილულია კარლემან-ბერს-ვეკუას განტოლებების ამონახსნთა სივრცეების ალგებრული სტრუქტურა. ბერსის აზრით მაღალი რიგის წარმოებულის გამოყენებით აგებულია განზოგადებულ ანალიზურ ფუნქციათა წრფივი სივრცეების (ნამდვილ რიცხვთა ღერძზე) მიმდევრობა და ამ გზით მოხდენილია ამონახსნთა სივრცის კლასიფიკაცია. პირველი გვარის ფსევდოანალიზური ფუნქციების პერიოდულობის თვისება გავრცელებულია მეორე გვარის ფუნქციებზე და ამ გზით დადგენილია ანალოგიური თვისებები ბელტრანის განტოლების ამონახსნებისათვის. განხილულია აგრეთვე მომიჯნავე ამოცანები და პრობლემები.

REFERENCES

1. Akhalaia G., Giorgadze G., Kaldani N., Jikia V., Manjavidze N., Makatsaria G. (2012) Elliptic systems on Riemnn surfaces. *Lecture Notes* of TICMI, 13: 3-112.
2. Vekua I. N. (1962) Generalized analytic functions, Pergamon Press.
3. Bers L. (1950) Theory of pseudo-analytic functions. New York University. N.Y.
4. Kravchenko V. (2009) Applied pseudoanalytic function theory. Birkhauser Verlag, Basel.
5. Protter M.H. (1956) The periodicity problem for pseudoanalytic functions. *Ann. of Math*., **64**, 2: 154-174.

Received September, 2018