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Mathematics

On Approximation of Three-Dimensional Model of Thermoelastic Piezoelectric Plates by Two-Dimensional Problems

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ABSTRACT. In the present paper initial-boundary value problem corresponding to three-dimensional model of thermoelastic piezoelectric solid consisting of anisotropic inhomogeneous material with regard to magnetic field is considered. An algorithm of approximation of the three-dimensional dynamical model by two-dimensional problems for plate with variable thickness is constructed, when density of surface force, and components of electric displacement and magnetic induction along the outward normal vector of the boundary are given on the upper and the lower face surfaces of the plate. The obtained two-dimensional initial-boundary value problems are investigated in suitable function spaces. Moreover, convergence of the sequence of vector-functions of three space variables restored from the solutions of the constructed two-dimensional problems to the solution of the original three-dimensional initial-boundary value problem is proved and under additional conditions the rate of convergence is estimated. *© 2018 Bull. Georg. Natl. Acad. Sci.*

Key words: piezoelectric thermoelastic solids, initial-boundary value problems, plates with variable thickness, dimensional reduction algorithms, error estimates

Piezoelectric plates are widely used in various engineering constructions and hence it is important to construct and investigate algorithms of approximation of three-dimensional models of plates by two-dimensional problems. Mathematical models of interaction between elastic, electric and thermal fields in piezoelectric body were developed by W. Voigt [1] and further various models of piezoelectric solids were constructed and investigated by various authors (see [2] and the references given therein). One of the methods of constructing two-dimensional models for plates with variable thickness in the classical theory of elasticity was suggested by I. Vekua in [3]. Various two-dimensional mode1s constructed by Vekua were collected in his monograph [4]. The estimates of the order of approximation of static three-dimensional problem by two-dimensional ones constructed in [3] first were obtained in the spaces of classical regular functions in the paper [5], and the reduced two-dimensional models for thin shallow shells constructed in

[4] were investigated in Sobolev spaces in [6]. Later on, dimensional reduction methods suggested by I. Vekua in [3, 4] and there generalizations were studied in the papers [7-11].

In this paper we construct and investigate an algorithm of approximation of three-dimensional dynamical model for thermoelastic piezoelectric plate with variable thickness, which may vanish on a part of the lateral boundary, by two-dimensional problems. We consider piezoelectric plate consisting of inhomogeneous anisotropic material, when temperature vanishes on the entire boundary, and applying variational approach we obtain the existence and uniqueness result for three-dimensional initial-boundary problem in suitable spaces of vector-valued distributions with values in Sobolev spaces. We construct a hierarchy of two-dimensional problems approximating the three-dimensional one, when densities of surface force and normal components of electric displacement and magnetic induction are given on the upper and the lower face surfaces of the plate. We investigate the constructed two-dimensional initial-boundary value problems in suitable function spaces. Moreover, we prove that the sequence of vector-functions of three space variables restored from the solutions of the constructed two-dimensional problems converges to the solution of the original three-dimensional problem and under additional regularity conditions we estimate the rate of convergence.

We denote by $W^{r,2}(D)=H^r(D)$ and $H^r(\hat{\Gamma})$, $r\in\mathbf{R}$, the Sobolev spaces of order r based on the spaces $H^0(D)=L^2(D)$ and $H^0(D)=L^2(D)$ of square-integrable functions, respectively, where $D\subset\mathbf{R}^p$, $p\in\mathbf{N}$, is a bounded Lipschitz domain [12] and $\hat{\Gamma}\subset\partial D$ is a Lipschitz surface. We denote by $\mathbf{H}^r(D)=[H^r(D)]^3$, $\mathbf{H}^r(\hat{\Gamma})=[H^r(\hat{\Gamma})]^3$, $\mathbf{L}^2(D)=[L^2(D)]^3$, $\mathbf{L}^s(\hat{\Gamma})=[L^s(\hat{\Gamma})]^3$, $s\ge 1$, $r,s\in\mathbf{R}$, the corresponding spaces of vector-valued functions. The trace operators we denote by $tr_{\hat{\Gamma}}:H^1(D)\to H^{1/2}(\hat{\Gamma})$ and $\mathbf{tr}_{\hat{\Gamma}}:\mathbf{H}^1(D)\to\mathbf{H}^{1/2}(\hat{\Gamma})$. For Banach space X, $C([0,T];X)$ denotes the space of continuous functions on $[0,T]$ with values in X, $L^q(0,T;X)$, $1\le q\le\infty$, is the space of such functions $g:(0,T)\to X$ that $\|g(t)\|_X\in L^q(0,T)$. We denote by $g'=dg/dt$ the generalized derivative of function $g\in L^q(0,T;X)$ [13].

Let us consider thermoelastic piezoelectric plate Ω with thickness vanishing on a part of its boundary, i.e. initial configuration of plate is a Lipschitz domain of the following form

$$\Omega=\left\{(x_1,x_2,x_3)\in\mathbf{R}^3;\, h^-(x_1,x_2)<x_3<h^+(x_1,x_2),\quad (x_1,x_2)\in\omega\right\},$$

where $\omega\subset\mathbf{R}^2$ is a two-dimensional bounded Lipschitz domain with boundary $\partial\omega$, $h^\pm\in C^0(\overline{\omega})\cap C^{0,1}_{loc}(\omega)$ are continuous on $\overline{\omega}$ and Lipschitz continuous in ω, $h^+(x_1,x_2)>h^-(x_1,x_2)$, for $(x_1,x_2)\in\omega\cup\tilde{\gamma}$, $\tilde{\gamma}\subset\partial\omega$ is a Lipschitz curve, $h^+(x_1,x_2)=h^-(x_1,x_2)$, for $(x_1,x_2)\in\partial\omega\setminus\tilde{\gamma}$. The upper and the lower faces surfaces of Ω defined by the equations $x_3=h^+(x_1,x_2)$ and $x_3=h^-(x_1,x_2)$, $(x_1,x_2)\in\omega$, we denote by Γ^+ and Γ^-, respectively, and the lateral surface, where the thickness of Ω is positive, we denote by $\tilde{\Gamma}=\partial\Omega\setminus(\overline{\Gamma^+\cup\Gamma^-})=\{(x_1,x_2,x_3)\in\mathbf{R}^3; h^-(x_1,x_2)<x_3<h^+(x_1,x_2),\ (x_1,x_2)\in\tilde{\gamma}\}$.

We assume that thermoelastic piezoelectric plate consists of general inhomogeneous anisotropic material with the mass density ρ in the initial configuration, the elasticity tensor $(c_{ijpq})_{i,j,p,q=1}^3$, the piezoelectric $(\varepsilon_{pij})_{i,j,p=1}^3$ and piezomagnetic $(b_{pij})_{i,j,p=1}^3$ coefficients, the stress-temperature tensor $(\lambda_{ij})_{i,j=1}^3$, the permittivity $(d_{ij})_{i,j=1}^3$ and permeability $(\zeta_{ij})_{i,j=1}^3$ tensors, the coupling coefficients connecting electric and magnetic fields $(a_{ij})_{i,j=1}^3$, the thermal conductivity tensor $(\eta_{ij})_{i,j=1}^3$ and the thermal capacity χ. We neglect the influence of thermal field on electric and magnetic fields and assume the coefficients characterizing the relation between thermal, and electric and magnetic fields vanish. The applied body force density we denote by $\mathbf{f}=(f_i)_{i=1}^3:\Omega\times(0,T)\to\mathbf{R}^3$, the density of electric charges we denote by $f^\varepsilon:\Omega\times(0,T)\to\mathbf{R}$, and the density of heat sources we denote by $f^\theta:\Omega\times(0,T)\to\mathbf{R}$. The temperature θ vanishes along the boundary $\Gamma=\partial\Omega$ of the domain Ω. The plate is clamped along a part $\tilde\Gamma_0=\{(x_1,x_2,x_3)\in\tilde\Gamma;\ (x_1,x_2)\in\tilde\gamma_0\}$, $\tilde\gamma_0\subset\tilde\gamma$, of the lateral surface $\tilde\Gamma$ and on the remaining part $\Gamma_1=\Gamma\setminus\overline{\tilde\Gamma_0}$ of the boundary surface force with density $\mathbf{g}=(g_i):\Gamma_1\times(0,T)\to\mathbf{R}^3$ is given. The electric potential φ vanishes along $\tilde\Gamma_0^\varphi=\{(x_1,x_2,x_3)\in\tilde\Gamma;\ (x_1,x_2)\in\tilde\gamma_0^\varphi\}$, $\tilde\gamma_0^\varphi\subset\tilde\gamma$, of the lateral surface $\tilde\Gamma$ and on the remaining part $\Gamma_1^\varphi=\Gamma\setminus\overline{\Gamma_0^\varphi}$ of the boundary the normal component of the electric displacement with density $g^\varphi:\Gamma_1^\varphi\times(0,T)\to\mathbf{R}$ is given. The magnetic potential ψ vanishes along $\tilde\Gamma_0^\psi=\{(x_1,x_2,x_3)\in\tilde\Gamma;\ (x_1,x_2)\in\tilde\gamma_0^\psi\}$, $\tilde\gamma_0^\psi\subset\tilde\gamma$, of the lateral surface $\tilde\Gamma$ and on the remaining part $\Gamma_1^\psi=\Gamma\setminus\overline{\Gamma_0^\psi}$ of the boundary the normal component of the magnetic induction with density $g^\psi:\Gamma_1^\psi\times(0,T)\to\mathbf{R}$ is given. The linear dynamical three-dimensional model of the thermoelastic piezoelectric body Ω in differential form with quasi-static equations for electro-magnetic fields, where the rate of change of magnetic field is small, i.e. electric field is curl free, and there is no electric current, i.e. magnetic field is curl free, is given by the following initial-boundary value problem [2]:

$$\rho\frac{\partial^2 u_i}{\partial t^2}-\sum_{j=1}^3\frac{\partial}{\partial x_j}\left(\sum_{p,q=1}^3 c_{ijpq}e_{pq}(\mathbf{u})+\sum_{p=1}^3\varepsilon_{pij}\frac{\partial\varphi}{\partial x_p}+\sum_{p=1}^3 b_{pij}\frac{\partial\psi}{\partial x_p}-\lambda_{ij}\theta\right)=f_i\quad\text{in }\Omega\times(0,T),\ i=1,2,3,\tag{1}$$

$$\sum_{i=1}^3\frac{\partial}{\partial x_i}\left(\sum_{p,q=1}^3\varepsilon_{ipq}e_{pq}(\mathbf{u})-\sum_{i=1}^3 d_{ij}\frac{\partial\varphi}{\partial x_j}-\sum_{j=1}^3 a_{ij}\frac{\partial\psi}{\partial x_j}\right)=f^\varepsilon\quad\text{in }\Omega\times(0,T),\tag{2}$$

$$\sum_{i=1}^3\frac{\partial}{\partial x_i}\left(\sum_{p,q=1}^3 b_{ipq}e_{pq}(\mathbf{u})-\sum_{j=1}^3 a_{ij}\frac{\partial\varphi}{\partial x_j}-\sum_{j=1}^3\zeta_{ij}\frac{\partial\psi}{\partial x_j}\right)=0\quad\text{in }\Omega\times(0,T),\tag{3}$$

$$\chi\frac{\partial\theta}{\partial t}-\sum_{j=1}^3\frac{\partial}{\partial x_i}\left(\eta_{ij}\frac{\partial\theta}{\partial x_j}\right)+\Theta_0\frac{\partial}{\partial t}\sum_{i,j=1}^3\lambda_{ij}e_{ij}(\mathbf{u})=f^\theta\quad\text{in }\Omega\times(0,T),\tag{4}$$

$$\mathbf{u}=\mathbf{0}\ \text{ on }\tilde\Gamma_0\times(0,T),\ \varphi=0\ \text{ on }\tilde\Gamma_0^\varphi\times(0,T),\ \psi=0\ \text{ on }\tilde\Gamma_0^\psi\times(0,T),\ \theta=0\ \text{ on }\Gamma\times(0,T),\tag{5}$$

$$\sum_{j=1}^3\left(\sum_{p,q=1}^3 c_{ijpq}e_{pq}(\mathbf{u})+\sum_{p=1}^3\varepsilon_{pij}\frac{\partial\varphi}{\partial x_p}+\sum_{p=1}^3 b_{pij}\frac{\partial\psi}{\partial x_p}-\lambda_{ij}\theta\right)n_j=g_i\quad\text{on }\Gamma_1\times(0,T),\ i=1,2,3,\tag{6}$$

$$\sum_{i=1}^3\left(\sum_{p,q=1}^3\varepsilon_{ipq}e_{pq}(\mathbf{u})-\sum_{i=1}^3 d_{ij}\frac{\partial\varphi}{\partial x_j}-\sum_{j=1}^3 a_{ij}\frac{\partial\psi}{\partial x_j}\right)n_i=g^\varphi\quad\text{on }\Gamma_1^\varphi\times(0,T),\tag{7}$$

$$\sum_{i=1}^3\left(\sum_{p,q=1}^3 b_{ipq}e_{pq}(\mathbf{u})-\sum_{j=1}^3 a_{ij}\frac{\partial\varphi}{\partial x_j}-\sum_{j=1}^3\zeta_{ij}\frac{\partial\psi}{\partial x_j}\right)n_i=g^\psi\quad\text{on }\Gamma_1^\psi\times(0,T),\tag{8}$$

$$u_i(x,0)=u_{0i}(x),\ \frac{\partial u_i}{\partial t}(x,0)=u_{1i}(x),\ \theta(x,0)=\theta_0(x)\quad\text{in }\Omega,\ i=1,2,3,\tag{9}$$

where $\mathbf{n}=(n_i)_{i=1}^3$ is the unit outward normal vector to Γ, $\mathbf{u}=(u_i):\Omega\times(0,T)\to\mathbf{R}^3$ is the displacement vector-function, $\varphi:\Omega\times(0,T)\to\mathbf{R}$ and $\psi:\Omega\times(0,T)\to\mathbf{R}$ stand for the electric and magnetic potentials such that the electric and magnetic fields are $\mathbf{E}=-grad\varphi$ and $\mathbf{H}=-grad\psi$, $\theta:\Omega\times(0,T)\to\mathbf{R}$ is the temperature distribution, $e_{ij}(\mathbf{v})=1/2\left(\partial v_i/\partial x_j+\partial v_j/\partial x_i\right)$, $i,j=1,2,3$, $\mathbf{v}=(v_i)_{i=1}^3$, is the strain tensor,

$\Theta_0 > 0$ is the temperature of thermoelastic piezoelectric body in natural state in the absence of deformation and electromagnetic fields, which is considered as a reference temperature, $\mathbf{u}_0 = (u_{0i})_{i=1}^3$ and $\mathbf{u}_1 = (u_{1i})_{i=1}^3$ are the initial displacement and velocity vector-functions, θ_0 is the initial distribution of temperature. We assume that the coefficients characterizing elastic, thermal, electric and magnetic properties satisfy the following symmetry conditions

$$\begin{aligned} &c_{ijpq}(x) = c_{ijqp}(x) = c_{jipq}(x), \quad \varepsilon_{pij}(x) = \varepsilon_{pji}(x), \quad b_{pij}(x) = b_{pji}(x), \\ &d_{ij}(x) = d_{ji}(x), \quad a_{ij}(x) = a_{ji}(x), \quad \zeta_{ij}(x) = \zeta_{ji}(x), \quad \lambda_{ij}(x) = \lambda_{ji}(x), \quad x \in \Omega, \ i,j,p,q = 1,2,3. \end{aligned} \tag{10}$$

To investigate the existence and uniqueness of solution of the three-dimensional initial-boundary value problem (1)-(9) we consider the following variational formulation, which is equivalent to the differential formulation in spaces of smooth enough functions: Find the unknown vector-function $\mathbf{u} \in C([0,T];\mathbf{V}(\Omega))$, $\mathbf{u}' \in C([0,T];\mathbf{L}^2(\Omega))$, and functions $\varphi \in C([0,T];V^{\varphi}(\Omega))$, $\psi \in C([0,T];V^{\psi}(\Omega))$, $\theta \in L^2(0,T;H_0^1(\Omega))$, $\theta' \in L^2(0,T;H^{-1}(\Omega))$, which satisfy the following equations in the sense of distributions on $(0,T)$,

$$\frac{d}{dt}(\rho\mathbf{u}',\mathbf{v})_{\mathbf{L}^2(\Omega)} + c(\mathbf{u},\mathbf{v}) + \varepsilon(\varphi,\mathbf{v}) + b(\psi,\mathbf{v}) - \lambda(\theta,\mathbf{v}) = L^{\mathbf{u}}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}(\Omega), \tag{11}$$

$$-\varepsilon(\bar{\varphi},\mathbf{u}) + d(\varphi,\bar{\varphi}) + a(\psi,\bar{\varphi}) = L^{\varphi}(\bar{\varphi}), \qquad \forall \bar{\varphi} \in V^{\varphi}(\Omega), \tag{12}$$

$$-b(\bar{\psi},\mathbf{u}) + a(\varphi,\bar{\psi}) + \zeta(\psi,\bar{\psi}) = L^{\psi}(\bar{\psi}), \qquad \forall \bar{\psi} \in V^{\psi}(\Omega), \tag{13}$$

$$\frac{d}{dt}(\chi\theta,\bar{\theta})_{L^2(\Omega)} + \eta(\theta,\bar{\theta}) + \Theta_0\lambda(\bar{\theta},\mathbf{u}') = L^{\theta}(\bar{\theta}), \qquad \forall \bar{\theta} \in H_0^1(\Omega), \tag{14}$$

together with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{u}_1, \quad \theta(0) = \theta_0, \tag{15}$$

where $\mathbf{V}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega); \ \ \mathbf{tr}_{\Gamma}(\mathbf{v}) = \mathbf{0} \ \text{ on } \ \Gamma_0\}$, $V^{\varphi}(\Omega) = \{\bar{\varphi} \in H^1(\Omega); \ \ tr_{\Gamma}(\bar{\varphi}) = 0 \ \text{ on } \ \Gamma_0^{\varphi}\}$, $V^{\psi}(\Omega) = \{\bar{\psi} \in H^1(\Omega); tr_{\Gamma}(\bar{\psi}) = 0 \ \text{ on } \ \Gamma_0^{\psi}\}$, $H_0^1(\Omega) = \{\bar{\theta} \in H^1(\Omega); tr_{\Gamma}(\bar{\theta}) = 0 \ \text{ on } \ \Gamma\}$,

$$c(\mathbf{u},\mathbf{v}) = \int_{\Omega} \sum_{i,j,p,q=1}^{3} c_{ijpq} e_{pq}(\mathbf{u}) e_{ij}(\mathbf{v}) dx, \ \ \varepsilon(\varphi,\mathbf{v}) = \int_{\Omega} \sum_{i,j,p=1}^{3} \varepsilon_{pij} \frac{\partial \varphi}{\partial x_p} e_{ij}(\mathbf{v}) dx,$$

$$b(\psi,\mathbf{v}) = \int_{\Omega} \sum_{i,j,p=1}^{3} b_{pij} \frac{\partial \psi}{\partial x_p} e_{ij}(\mathbf{v}) dx, \ \ \lambda(\theta,\mathbf{v}) = -\int_{\Omega} \sum_{i,j=1}^{3} v_i \frac{\partial}{\partial x_j}(\lambda_{ij}\theta) dx, \ \ d(\varphi,\bar{\varphi}) = \int_{\Omega} \sum_{i,j=1}^{3} d_{ij} \frac{\partial \varphi}{\partial x_j}\frac{\partial \bar{\varphi}}{\partial x_i} dx,$$

$$a(\psi,\bar{\varphi}) = \int_{\Omega} \sum_{i,j=1}^{3} a_{ij} \frac{\partial \psi}{\partial x_j}\frac{\partial \bar{\varphi}}{\partial x_i} dx, \ \ \zeta(\psi,\bar{\psi}) = \int_{\Omega} \sum_{i,j=1}^{3} \zeta_{ij} \frac{\partial \psi}{\partial x_j}\frac{\partial \bar{\psi}}{\partial x_i} dx, \ \ \eta(\theta,\bar{\theta}) = \int_{\Omega} \sum_{i,j=1}^{3} \eta_{ij} \frac{\partial \theta}{\partial x_j}\frac{\partial \bar{\theta}}{\partial x_i} dx,$$

$$L^{\mathbf{u}}(\mathbf{v}) = \int_{\Omega} \sum_{i=1}^{3} f_i v_i dx + \int_{\Gamma_1} \sum_{i=1}^{3} g_i tr_{\Gamma_1}(v_i) d\Gamma, \ \ L^{\varphi}(\bar{\varphi}) = \int_{\Omega} f^{\varepsilon}\bar{\varphi} dx - \int_{\Gamma_1^{\varphi}} g^{\varphi} tr_{\Gamma_1^{\varphi}}(\bar{\varphi}) d\Gamma,$$

$$L^{\psi}(\bar{\psi}) = -\int_{\Gamma_1^{\psi}} g^{\psi} tr_{\Gamma_1^{\psi}}(\bar{\psi}) d\Gamma, \ \ L^{\theta}(\bar{\theta}) = \int_{\Omega} f^{\theta}\bar{\theta} dx,$$

$(.,.)_{\mathbf{L}^2(\Omega)}$ and $(.,.)_{L^2(\Omega)}$ are the scalar products in the spaces $\mathbf{L}^2(\Omega)$ and $L^2(\Omega)$, respectively.

For the initial-boundary value problem (11)-(15) corresponding to the dynamical three-dimensional model for thermoelastic piezoelectric plate with regard to magnetic field the following theorem is valid.

Theorem 1. *Suppose that $\Omega \subset \mathbf{R}^3$ is a bounded domain with Lipschitz boundary, $\tilde{\Gamma}_0^{\varphi} \neq \varnothing$, $\tilde{\Gamma}_0^{\psi} \neq \varnothing$, and the parameters characterizing thermo-elastic and electro-magnetic properties of the body Ω are such that $\rho, \chi \in L^{\infty}(\Omega)$, $\rho(x) > \alpha_{\rho} = const > 0$, $\chi(x) > \alpha_{\chi} = const > 0$, for almost all $x \in \Omega$, c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, η_{ij}, $\lambda_{ij} \in L^{\infty}(\Omega)$, $\partial\lambda_{ij}/\partial x_j \in L^3(\Omega)$, $i,j,p,q = 1,2,3$, and satisfy symmetry conditions* (10) *and the following positive definiteness conditions*

$$\sum_{i,j,p,q=1}^{3} c_{ijpq}\xi_{ij}\xi_{pq} \geq \alpha_c \sum_{i,j=1}^{3}(\xi_{ij})^2, \ \forall \xi_{ij} \in \mathbf{R}, \xi_{ij} = \xi_{ji}, \quad \sum_{i,j=1}^{3}\eta_{ij}\xi_j\xi_j \geq \alpha_\eta \sum_{i=1}^{3}(\xi_i)^2, \ \forall \xi_i \in \mathbf{R},$$
$$\sum_{i,j=1}^{3} d_{ij}\xi_j\xi_i + 2\sum_{i,j=1}^{3} a_{ij}\overline{\xi}_j\xi_i + \sum_{i,j=1}^{3}\zeta_{ij}\overline{\xi}_j\overline{\xi}_i \geq \alpha\sum_{i=1}^{3}\left((\xi_i)^2 + (\overline{\xi}_i)^2\right), \quad \forall \xi_i, \overline{\xi}_i \in \mathbf{R}, \tag{16}$$

for almost all $x \in \Omega$, where α_c, α_η, α are positive constants. If $\mathbf{u}_0 \in \mathbf{V}(\Omega)$, $\mathbf{u}_1 \in \mathbf{L}^2(\Omega)$, $\theta_0 \in L^2(\Omega)$, $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g}, \mathbf{g}' \in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, $f^\varepsilon, (f^\varepsilon)' \in L^2(0,T;L^{6/5}(\Omega))$, $g^\varphi, (g^\varphi)' \in L^2(0,T;$ $L^{4/3}(\Gamma_1^\varphi))$, $g^\psi, (g^\psi)' \in L^2(0,T;L^{4/3}(\Gamma_1^\psi))$, $f^\theta \in L^2(0,T;L^{6/5}(\Omega))$, then the initial-boundary value problem (11)-(15) possesses a unique solution.

In order to construct the hierarchy of two-dimensional models let us consider the subspaces $\mathbf{V}_\mathbf{N}(\Omega)$ and $\mathbf{H}_\mathbf{N}(\Omega)$ of $\mathbf{V}(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively, $\mathbf{N} = (N_1, N_2, N_3)$, consisting of vector-functions whose components are polynomials with respect to the variable x_3,

$$\mathbf{v_N} = (v_{\mathbf{N}i}), \ v_{\mathbf{N}i} = \sum_{r_i=0}^{N_i}\frac{1}{h}\left(r_i + \frac{1}{2}\right)\overset{r_i}{v}_{\mathbf{N}i} P_{r_i}(z), \quad \overset{r_i}{v}_{\mathbf{N}i} \in L^2(\omega), \quad 0 \leq r_i \leq N_i, \ i = 1,2,3,$$

where $z = \dfrac{x_3 - \overline{h}}{h}$, $h = \dfrac{h^+ - h^-}{2}$, $\overline{h} = \dfrac{h^+ + h^-}{2}$. We also consider the subspaces $V^\varphi_{N_\varphi}(\Omega)$ and $V^\psi_{N_\psi}(\Omega)$ of $V^\varphi(\Omega)$ and $V^\psi(\Omega)$, respectively, which consist of the following functions

$$\overline{\varphi}_{N_\varphi} = \sum_{r_\varphi=0}^{N_\varphi}\frac{1}{h}\left(r_\varphi + \frac{1}{2}\right)\overset{r_\varphi}{\overline{\varphi}}_{N_\varphi} P_{r_\varphi}(z), \quad \overset{r_\varphi}{\overline{\varphi}}_{N_\varphi} \in L^2(\omega), \ r_\varphi = 0,\dots,N_\varphi,$$
$$\overline{\psi}_{N_\psi} = \sum_{r_\psi=0}^{N_\psi}\frac{1}{h}\left(r_\psi + \frac{1}{2}\right)\overset{r_\psi}{\overline{\psi}}_{N_\psi} P_{r_\psi}(z), \quad \overset{r_\psi}{\overline{\psi}}_{N_\psi} \in L^2(\omega), \ r_\psi = 0,\dots,N_\psi,$$

and subspaces $V^\theta_{N_\theta}(\Omega)$ and $H^\theta_{N_\theta}(\Omega)$ of $H^1_0(\Omega)$ and $L^2(\Omega)$, consisting of the functions

$$\overline{\theta}_{N_\theta} = \sum_{r_\theta=0}^{N_\theta}\frac{1}{h}\left(r_\theta + \frac{1}{2}\right)\overset{r_\theta}{\overline{\theta}}_{N_\theta} P_{r_\theta}(z) - \frac{1}{2}\sum_{r=0}^{N_\theta}\frac{1}{h}(1-(-1)^{r_\theta+N_\theta})\left(r_\theta + \frac{1}{2}\right)\overset{r_\theta}{\overline{\theta}}_{N_\theta} P_{N_\theta+1}(z)$$
$$-\frac{1}{2}\sum_{r_\theta=0}^{N_\theta}\frac{1}{h}(1+(-1)^{r_\theta+N_\theta})\left(r_\theta + \frac{1}{2}\right)\overset{r_\theta}{\overline{\theta}}_{N_\theta} P_{N_\theta+2}(z), \quad \overset{r_\theta}{\overline{\theta}}_{N_\theta} \in L^2(\omega), \ r_\theta = 0,\dots,N_\theta.$$

Since the functions h^+ and h^- are Lipschitz continuous in ω from Rademacher's theorem [14], we have that h^+ and h^- are differentiable almost everywhere in ω and $\partial_\alpha h^\pm \in L^\infty(\omega^*)$ for all subdomains ω^*, $\overline{\omega^*} \subset \omega$, $\alpha = 1,2$. Therefore, the positiveness of h in ω implies that for any vector-function $\mathbf{v_N} = (v_{\mathbf{N}i})_{i=1}^3 \in \mathbf{V_N}(\Omega)$ the corresponding functions $\overset{r_i}{v}_{\mathbf{N}i} \in H^1(\omega^*)$, for all ω^*, $\overline{\omega^*} \subset \omega$, i.e. $\overset{r_i}{v}_{\mathbf{N}i} \in H^1_{loc}(\omega)$, $0 \leq r_i \leq N_i$, $i = 1,2,3$. Similarly, for all functions $\overline{\varphi}_{N_\varphi} \in V^\varphi_{N_\varphi}(\Omega)$, $\overline{\psi}_{N_\psi} \in V^\psi_{N_\psi}(\Omega)$, $\overline{\theta}_{N_\theta} \in V^\theta_{N_\theta}(\Omega)$, the functions $\overset{r_\varphi}{\overline{\varphi}}_{N_\varphi}$, $\overset{r_\psi}{\overline{\psi}}_{N_\psi}$, $\overset{r_\theta}{\overline{\theta}}_{N_\theta}$ of two space variables in the expressions of $\overline{\varphi}_{N_\varphi}, \overline{\psi}_{N_\psi}$, $\overline{\theta}_{N_\theta}$ belong to $H^1(\omega^*)$, $\overline{\omega^*} \subset \omega$, i.e. $\overset{r_\varphi}{\overline{\varphi}}_{N_\varphi}$, $\overset{r_\psi}{\overline{\psi}}_{N_\psi}$, $\overset{r_\theta}{\overline{\theta}}_{N_\theta} \in H^1_{loc}(\omega)$, $r_\varphi = 0,\dots,N_\varphi$, $r_\psi = 0,\dots,N_\psi$, $r_\theta = 0,\dots,N_\theta$. Moreover, the norms $\|.\|_{\mathbf{H}^1(\Omega)}$ and $\|.\|_{H^1(\Omega)}$ in the spaces $\mathbf{H}^1(\Omega)$ and $H^1(\Omega)$ define weighted norms $\|.\|_*$ and $\|.\|_{\varphi*}$, $\|.\|_{\psi*}$, $\|.\|_{\theta*}$ of vector-functions $\vec{v}_\mathbf{N} = (\overset{r_i}{v}_{\mathbf{N}i}) \in [H^1_{loc}(\omega)]^{N_{1,2,3}}$, $N_{1,2,3} = N_1 + N_2 + N_3 + 3$, and $\vec{\overline{\varphi}}_{N_\phi} = (\overset{r_\phi}{\overline{\varphi}}_{N_\phi}) \in [H^1_{loc}(\omega)]^{N_\varphi+1}$, $\vec{\overline{\psi}}_{N_\psi} = (\overset{r_\psi}{\overline{\psi}}_{N_\psi}) \in [H^1_{loc}(\omega)]^{N_\psi+1}$, $\vec{\overline{\theta}}_{N_\theta} = (\overset{r_\theta}{\overline{\theta}}_{N_\theta}) \in [H^1_{loc}(\omega)]^{N_\theta+1}$, such that $\|\vec{v}_\mathbf{N}\|_* = \|\mathbf{v_N}\|_{\mathbf{H}^1(\Omega)}$ and $\|\vec{\overline{\varphi}}_{N_\varphi}\|_{\varphi*} = \|\overline{\varphi}_{N_\varphi}\|_{H^1(\Omega)}$, $\|\vec{\overline{\psi}}_{N_\psi}\|_{\psi*} = \|\overline{\psi}_{N_\psi}\|_{H^1(\Omega)}$, $\|\vec{\overline{\theta}}_{N_\theta}\|_{\theta*} = \|\overline{\theta}_{N_\theta}\|_{H^1(\Omega)}$. Using the properties of the

Legendre polynomials [15], we can obtain explicit expressions of the norms $\|.\|_*$ and $\|.\|_{\varphi*}, \|.\|_{\psi*}, \|.\|_{\theta*}$. In particular, $\|.\|_*$ is given by the following expression:

$$\|\vec{v}_{\mathbf{N}}\|_*^2 = \sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\left(r_i+\frac{1}{2}\right)\left[\left\|\sum_{s_i=r_i}^{N_i}\left(s_i+\frac{1}{2}\right)(1-(-1)^{r_i+s_i})h^{-3/2}\overset{s_i}{v}_{\mathbf{N}i}\right\|_{L^2(\omega)}^2 + \|h^{-1/2}\overset{r_i}{v}_{\mathbf{N}i}\|_{L^2(\omega)}^2 + \sum_{\alpha=1}^{2}\left\|\sum_{s_i=r_i+1}^{N_i}\left(s_i+\frac{1}{2}\right)\right.\right.$$
$$\left.\left.(\partial_\alpha h^+ - (-1)^{r_i+s_i}\partial_\alpha h^-)h^{-3/2}\overset{s_i}{v}_{\mathbf{N}i} - h^{-1/2}\partial_\alpha \overset{r_i}{v}_{\mathbf{N}i} + (r_i+1)h^{-3/2}\partial_\alpha h \overset{r_i}{v}_{\mathbf{N}i}\right\|_{L^2(\omega)}^2\right].$$

For components $\overset{r_i}{v}_{\mathbf{N}i}$ and $\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}$, $\overset{r_\psi}{\bar{\psi}}_{N_\psi}$, $\overset{r_\theta}{\bar{\theta}}_{N_\theta}$ of $\vec{v}_{\mathbf{N}}$ and $\vec{\bar{\varphi}}_{N_\varphi}$, $\vec{\bar{\psi}}_{N_\psi}$, $\vec{\bar{\theta}}_{N_\theta}$, which possess the properties $\|\vec{v}_{\mathbf{N}}\|_* < \infty$ and $\|\vec{\bar{\varphi}}_{N_\varphi}\|_{\varphi*} < \infty$, $\|\vec{\bar{\psi}}_{N_\psi}\|_{\psi*} < \infty$, $\|\vec{\bar{\theta}}_{N_\theta}\|_{\theta*} < \infty$ we can define the traces on $\tilde{\gamma}$. Indeed, the corresponding vector-function of three space variables $\mathbf{v}_{\mathbf{N}} = (v_{\mathbf{N}i})_{i=1}^3$ and functions $\bar{\varphi}_{N_\varphi}, \bar{\psi}_{N_\psi}$, $\bar{\theta}_{N_\theta}$ belong to the space $\mathbf{V}_{\mathbf{N}}(\Omega) \subset \mathbf{H}^1(\Omega)$ and $V_{N_\varphi}^{\varphi}(\Omega)$, $V_{N_\psi}^{\psi}(\Omega)$, $V_{N_\theta}^{\theta}(\Omega) \subset H^1(\Omega)$, respectively. Consequently, we can define the traces of $\overset{r_i}{v}_{\mathbf{N}i}, \overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}$, $\overset{r_\psi}{\bar{\psi}}_{N_\psi}$, $\overset{r_\theta}{\bar{\theta}}_{N_\theta}$ on $\tilde{\gamma}$,

$$tr_{\tilde{\gamma}}(\overset{r_i}{v}_{\mathbf{N}i}) = \int_{h^-}^{h^+} tr_{\tilde{\Gamma}}(v_{\mathbf{N}i})P_{r_i}(z)dx_3,\ r_i = 0,..,N_i, i=1,2,3,\quad tr_{\tilde{\gamma}}(\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}) = \int_{h^-}^{h^+} tr_{\tilde{\Gamma}}(\bar{\varphi}_{N_\varphi})P_{r_\varphi}(z)dx_3,\ r_\varphi = 0,...,N_\varphi,$$

$$tr_{\tilde{\gamma}}(\overset{r_\psi}{\bar{\psi}}_{N_\psi}) = \int_{h^-}^{h^+} tr_{\tilde{\Gamma}}(\bar{\psi}_{N_\psi})P_{r_\psi}(z)dx_3,\ r_\psi = 0,...,N_\psi,\ tr_{\tilde{\gamma}}(\overset{r_\theta}{\bar{\theta}}_{N_\theta}) = \int_{h^-}^{h^+} tr_{\tilde{\Gamma}}(\bar{\theta}_{N_\theta})P_{r_\theta}(z)dx_3,\ r_\theta = 0,...,N_\theta.$$

Since the vector-functions $\mathbf{v}_{\mathbf{N}}$ from the subspaces $\mathbf{V}_{\mathbf{N}}(\Omega)$ and $\mathbf{H}_{\mathbf{N}}(\Omega)$, and the functions $\bar{\varphi}_{N_\varphi} \in V_{N_\varphi}^{\varphi}(\Omega)$, $\bar{\psi}_{N_\psi} \in V_{N_\psi}^{\psi}(\Omega)$, and $\bar{\theta}_{N_\theta}$ from $V_{N_\theta}^{\theta}(\Omega)$ and $H_{N_\theta}^{\theta}(\Omega)$ are uniquely defined by the functions $\overset{r_i}{v}_{\mathbf{N}i}$, $\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}$, $\overset{r_\psi}{\bar{\psi}}_{N_\psi}$, $\overset{r_\theta}{\bar{\theta}}_{N_\theta}$ of two space variables, hence considering the original three-dimensional problem (11)-(15) on these subspaces, we obtain the following hierarchy of two-dimensional initial-boundary value problems: Find $\vec{u}_{\mathbf{N}} \in C([0,T];\vec{V}_{\mathbf{N}}(\omega))$, $\vec{u}'_{\mathbf{N}} \in C([0,T];\vec{H}_{\mathbf{N}}(\omega))$, $\vec{\varphi}_{N_\varphi} \in C([0,T];\vec{V}_{N_\varphi}^{\varphi}(\omega))$, $\vec{\psi}_{N_\psi} \in C([0,T];$ $\vec{V}_{N_\psi}^{\psi}(\omega))$, $\vec{\theta}_{N_\theta} \in L^2(0,T;\vec{V}_{N_\theta}^{\theta}(\omega))$, $\vec{\theta}'_{N_\theta} \in L^2(0,T;(\vec{V}_{N_\theta}^{\theta}(\omega))')$, which satisfy the following equations in the sense of distributions on $(0,T)$,

$$\frac{d}{dt}R_{\mathbf{N}}(\vec{u}'_{\mathbf{N}},\vec{v}_{\mathbf{N}}) + c_{\mathbf{N}}(\vec{u}_{\mathbf{N}},\vec{v}_{\mathbf{N}}) + \varepsilon_{N_\varphi\mathbf{N}}(\vec{\varphi}_{N_\varphi},\vec{v}_{\mathbf{N}}) + b_{N_\psi\mathbf{N}}(\vec{\psi}_{N_\psi},\vec{v}_{\mathbf{N}}) - \lambda_{N_\theta\mathbf{N}}(\vec{\theta}_{N_\theta},\vec{v}_{\mathbf{N}}) = L_{\mathbf{N}}^{\mathbf{u}}(\vec{v}_{\mathbf{N}}), \tag{17}$$

$$-\varepsilon_{N_\varphi\mathbf{N}}(\vec{\bar{\varphi}}_{N_\varphi},\vec{u}_{\mathbf{N}}) + d_{N_\varphi}(\vec{\varphi}_{N_\varphi},\vec{\bar{\varphi}}_{N_\varphi}) + a_{N_\varphi N_\psi}(\vec{\psi}_{N_\psi},\vec{\bar{\varphi}}_{N_\varphi}) = L_{N_\varphi}^{\varphi}(\vec{\bar{\varphi}}_{N_\varphi}), \tag{18}$$

$$-b_{N_\psi\mathbf{N}}(\vec{\bar{\psi}}_{N_\psi},\vec{u}_{\mathbf{N}}) + a_{N_\varphi N_\psi}(\vec{\varphi}_{N_\varphi},\vec{\bar{\psi}}_{N_\psi}) + \zeta_{N_\psi}(\vec{\psi}_{N_\psi},\vec{\bar{\psi}}_{N_\psi}) = L_{N_\psi}^{\psi}(\vec{\bar{\psi}}_{N_\psi}), \tag{19}$$

$$\frac{d}{dt}R_{N_\theta}^{\theta}(\vec{\theta}_{N_\theta},\vec{\bar{\theta}}_{N_\theta}) + \eta_{N_\theta}(\vec{\theta}_{N_\theta},\vec{\bar{\theta}}_{N_\theta}) + \Theta_0\lambda_{N_\theta\mathbf{N}}(\vec{\bar{\theta}}_{N_\theta},\vec{u}'_{\mathbf{N}}) = L_{N_\theta}^{\theta}(\vec{\bar{\theta}}_{N_\theta}), \tag{20}$$

for all $\vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega)$, $\vec{\bar{\varphi}}_{N_\varphi} \in V^{\varphi}(\Omega)$, $\vec{\bar{\psi}}_{N_\psi} \in \vec{V}_{N_\psi}^{\psi}(\omega)$, $\vec{\bar{\theta}}_{N_\theta} \in \vec{V}_{N_\theta}^{\theta}(\omega)$, and the initial conditions

$$\vec{u}_{\mathbf{N}}(0) = \vec{u}_{\mathbf{N}0},\quad \vec{u}'_{\mathbf{N}}(0) = \vec{u}_{\mathbf{N}1},\ \vec{\theta}_{N_\theta}(0) = \vec{\theta}_{N_\theta 0}, \tag{21}$$

where $\vec{V}_{\mathbf{N}}(\omega) = \{\vec{v}_{\mathbf{N}} = (\overset{r_i}{v}_{\mathbf{N}i}) \in [H_{loc}^1(\omega)]^{N_{1,2,3}}; \|\vec{v}_{\mathbf{N}}\|_* < \infty,\ tr_{\tilde{\gamma}}(\overset{r_i}{v}_{\mathbf{N}i}) = 0$ on $\tilde{\gamma}_0, r_i = 0,...,N_i, i=1,2,3\}$, $\vec{H}_{\mathbf{N}}(\omega) =$ $\{\vec{v}_{\mathbf{N}} = (\overset{r_i}{v}_{\mathbf{N}i}) \in [L^2(\omega)]^{N_{1,2,3}}; \|\vec{v}_{\mathbf{N}}\|_{\vec{H}_{\mathbf{N}}(\omega)}^2 = \sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\left\|h^{-1/2}\overset{r_i}{v}_{\mathbf{N}i}\right\|_{L^2(\omega)}^2 < \infty\}$, $\vec{V}_{N_\varphi}^{\varphi}(\omega) = \{\vec{\bar{\varphi}}_{N_\varphi} = (\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}) \in [H_{loc}^1(\omega)]^{N_\varphi+1};$ $\left\|\vec{\bar{\varphi}}_{N_\varphi}\right\|_{\varphi*} < \infty,\ tr_{\tilde{\gamma}}(\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}) = 0$ on $\tilde{\gamma}_0^{\varphi}, r_\varphi = 0,...,N_\varphi\}$, $\vec{V}_{N_\psi}^{\psi}(\omega) = \{\vec{\bar{\psi}}_{N_\psi} = (\overset{r_\psi}{\bar{\psi}}_{N_\psi}) \in [H_{loc}^1(\omega)]^{N_\psi+1}; \left\|\vec{\bar{\psi}}_{N_\psi}\right\|_{\psi*} < \infty,$ $tr_{\tilde{\gamma}}(\overset{r_\psi}{\bar{\psi}}_{N_\psi}) = 0$ on $\tilde{\gamma}_0^{\psi}, r_\psi = 0,...,N_\psi\}$, $\vec{V}_{N_\theta}^{\theta}(\omega) = \{\vec{\bar{\theta}}_{N_\theta} = (\overset{r_\theta}{\bar{\theta}}_{N_\theta}) \in [H_{loc}^1(\omega)]^{N_\theta+1}; \left\|\vec{\bar{\theta}}_{N_\theta}\right\|_{\theta*} < \infty,$

$tr_{\tilde{\gamma}}(\overset{r_\theta}{\bar{\theta}}_{N_\theta}) = 0$ on $\tilde{\gamma}_0^\theta$, $\quad r_\theta = 0,...,N_\theta\}$, $\quad \vec{H}^\theta_{N_\theta}(\omega) = \{\vec{\bar{\theta}}_{N_\theta} = (\overset{r_\theta}{\bar{\theta}}_{N_\theta}) \in [L^2(\omega)]^{N_\theta+1}; \left\|\vec{\bar{\theta}}_{N_\theta}\right\|^2_{\vec{H}^\theta_{N_\theta}(\omega)} =$ $\sum_{r_\theta=0}^{N_\theta}\left\|h^{-1/2}\overset{r_\theta}{\bar{\theta}}_{N_\theta}\right\|^2_{L^2(\omega)} < \infty\}$, the bilinear forms $R_{\mathbf{N}}$, $c_{\mathbf{N}}$, $\varepsilon_{N_\varphi \mathbf{N}}$, $b_{N_\psi \mathbf{N}}$, $\lambda_{N_\theta \mathbf{N}}$, d_{N_φ} $a_{N_\varphi N_\psi}$, ζ_{N_ψ}, $R^\theta_{N_\theta}$, η_{N_θ} are defined as follows $R_{\mathbf{N}}(\vec{\tilde{v}}_{\mathbf{N}}, \vec{v}_{\mathbf{N}}) = (\rho \tilde{\mathbf{v}}_{\mathbf{N}}, \mathbf{v}_{\mathbf{N}})_{\mathbf{L}^2(\Omega)}$, $c_{\mathbf{N}}(\vec{\tilde{v}}_{\mathbf{N}}, \vec{v}_{\mathbf{N}}) = c(\tilde{\mathbf{v}}_{\mathbf{N}}, \mathbf{v}_{\mathbf{N}})$, $\varepsilon_{N_\varphi \mathbf{N}}(\vec{\bar{\varphi}}_{N_\varphi}, \vec{v}_{\mathbf{N}}) = \varepsilon(\bar{\varphi}_{N_\varphi}, \mathbf{v}_{\mathbf{N}})$, $b_{N_\psi \mathbf{N}}(\vec{\bar{\psi}}_{N_\psi}, \vec{v}_{\mathbf{N}}) = b(\bar{\psi}_{N_\psi}, \mathbf{v}_{\mathbf{N}})$, $\lambda_{N_\theta \mathbf{N}}(\vec{\bar{\theta}}_{N_\theta}, \vec{v}_{\mathbf{N}}) = \lambda(\bar{\theta}_{N_\theta}, \mathbf{v}_{\mathbf{N}})$, $d_{N_\varphi}(\vec{\tilde{\varphi}}_{N_\varphi}, \vec{\bar{\varphi}}_{N_\varphi}) = d(\tilde{\varphi}_{N_\varphi}, \bar{\varphi}_{N_\varphi})$, $a_{N_\varphi N_\psi}(\vec{\bar{\varphi}}_{N_\varphi}, \vec{\bar{\psi}}_{N_\psi}) = a(\bar{\varphi}_{N_\varphi}, \bar{\psi}_{N_\psi})$, $\zeta_{N_\psi}(\vec{\tilde{\psi}}_{N_\psi}, \vec{\bar{\psi}}_{N_\psi}) = \zeta(\tilde{\psi}_{N_\psi}, \bar{\psi}_{N_\psi})$, $R^\theta_{N_\theta}(\vec{\tilde{\theta}}_{N_\theta}, \vec{\bar{\theta}}_{N_\theta}) = (\chi \tilde{\theta}_{N_\theta}, \bar{\theta}_{N_\theta})_{L^2(\Omega)}$, $\eta_{N_\theta}(\vec{\tilde{\theta}}_{N_\theta}, \vec{\bar{\theta}}_{N_\theta}) = \eta(\tilde{\theta}_{N_\theta}, \bar{\theta}_{N_\theta})$, for all vector-functions $\vec{\tilde{v}}_{\mathbf{N}}, \vec{v}_{\mathbf{N}} \in \vec{V}_{\mathbf{N}}(\omega)$, $\vec{\tilde{\varphi}}_{N_\phi}, \vec{\bar{\varphi}}_{N_\phi} \in \vec{V}^\varphi_{N_\varphi}(\omega)$, $\vec{\tilde{\psi}}_{N_\psi}$, $\vec{\bar{\psi}}_{N_\psi} \in \vec{V}^\psi_{N_\psi}(\omega)$, $\vec{\tilde{\theta}}_{N_\theta}, \vec{\bar{\theta}}_{N_\theta} \in \vec{V}^\theta_{N_\theta}(\omega)$, corresponding to $\tilde{\mathbf{v}}_{\mathbf{N}}, \mathbf{v}_{\mathbf{N}} \in \mathbf{V}_{\mathbf{N}}(\Omega)$, $\tilde{\phi}_{N_\phi}, \bar{\phi}_{N_\phi} \in V^\phi_{N_\phi}(\Omega)$, $\tilde{\psi}_{N_\psi}, \bar{\psi}_{N_\psi} \in V^\psi_{N_\psi}(\Omega)$, $\tilde{\theta}_{N_\theta}, \bar{\theta}_{N_\theta} \in V^\theta_{N_\theta}(\Omega)$, respectively. The linear forms $L^{\mathbf{u}}_{\mathbf{N}}, L^\varphi_{N_\varphi}, L^\psi_{N_\psi}$ and $L^\theta_{N_\theta}$ are given by the following expressions:

$$L^{\mathbf{u}}_{\mathbf{N}}(\vec{v}_{\mathbf{N}}) = \sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\left(r_i + \frac{1}{2}\right)\left[\int_\omega \frac{1}{h}\overset{r_i}{v}_{\mathbf{N}i}\left(\overset{r_i}{f}_i + g_i\big|_{\Gamma^+}\lambda_+ + g_i\big|_{\Gamma^-}\lambda_-(-1)^{r_i}\right)d\omega + \int_{\gamma_1}\frac{1}{h}tr_{\tilde{\gamma}}(\overset{r_i}{v}_{\mathbf{N}i})\overset{r_i}{g}_i\, d\gamma_1\right],$$

$$L^\varphi_{N_\varphi}(\vec{\bar{\varphi}}_{N_\varphi}) = \sum_{r_\varphi=0}^{N_\varphi}\left(r_\varphi + \frac{1}{2}\right)\left[\int_\omega \frac{1}{h}\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}\left(\overset{r_\varphi}{f^\varepsilon} + g^\varphi\big|_{\Gamma^+}\lambda_+ + g^\varphi\big|_{\Gamma^-}\lambda_-(-1)^{r_\varphi}\right)d\omega + \int_{\gamma_1^\varphi}\frac{1}{h}tr_{\tilde{\gamma}}(\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi})\overset{r_\varphi}{g^\varphi}\, d\gamma_1^\varphi\right],$$

$$L^\psi_{N_\psi}(\vec{\bar{\psi}}_{N_\psi}) = \sum_{r_\psi=0}^{N_\psi}\left(r_\psi + \frac{1}{2}\right)\left[\int_\omega \frac{1}{h}\overset{r_\psi}{\bar{\psi}}_{N_\psi}\left(g^\psi\big|_{\Gamma^+}\lambda_+ + g^\psi\big|_{\Gamma^-}\lambda_-(-1)^{r_\psi}\right)d\omega + \int_{\gamma_1^\psi}\frac{1}{h}tr_{\tilde{\gamma}}(\overset{r_\psi}{\bar{\psi}}_{N_\psi})\overset{r_\psi}{g^\psi}\, d\gamma_1^\psi\right],$$

$$L^\theta_{N_\theta}(\vec{\bar{\theta}}_{N_\theta}) = \sum_{r_\theta=0}^{N_\theta}\left(r_\theta + \frac{1}{2}\right)\int_\omega \frac{1}{h}\overset{r_\theta}{\bar{\theta}}_{N_\theta}\left(\overset{r_\theta}{f^\theta} - \sum_{\alpha=1}^{2}\frac{2(\alpha-1)-(-1)^\alpha}{2}((-1)^{r_\theta+N_\theta+\alpha}+1)\overset{N_\theta+\alpha}{f^\theta}\right)d\omega,$$

where $\gamma_1 = \tilde{\gamma}\setminus\tilde{\gamma}_0, \gamma_1^\varphi = \tilde{\gamma}\setminus\tilde{\gamma}_0^\varphi, \gamma_1^\psi = \tilde{\gamma}\setminus\tilde{\gamma}_0^\psi, \gamma_1^\theta = \tilde{\gamma}\setminus\tilde{\gamma}_0^\theta, \lambda_\pm = \sqrt{1+(\partial_1 h^\pm)^2 + (\partial_2 h^\pm)^2}$, $\overset{r}{v} = \int_{h^-}^{h^+} v P_r(z)dx_3$, for all functions $v \in L^2(\Omega)$, $r \in \mathbf{N}\cup\{0\}$.

For the constructed two-dimensional initial-boundary value problem (17)-(21) for thermoelastic piezoelectric plates the following existence and uniqueness theorem is proved.

Theorem 2. *Suppose that two-dimensional domain* ω *and functions* h^+, h^- *are such that* $\Omega\subset\mathbf{R}^3$ *is a Lipschitz domain,* $\tilde{\gamma}_0^\varphi \neq \varnothing$, $\tilde{\gamma}_0^\psi \neq \varnothing$, $\rho, \chi \in L^\infty(\Omega)$, $\rho(x) > \alpha_\rho = const > 0$, $\chi(x) > \alpha_\chi = const > 0$, *for almost all* $x\in\Omega$, c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, η_{ij}, $\lambda_{ij} \in L^\infty(\Omega)$, $\partial\lambda_{ij}/\partial x_j \in L^3(\Omega)$, $i,j,p,q = 1,2,3$, *and conditions* (10), (16) *are fulfilled. If*

$$h^{-1/2}\overset{r_i}{f}_i \in L^2(0,T;L^2(\omega)),\quad \lambda_\pm^{3/4}\frac{d^\alpha}{dt^\alpha}g_i\big|_{\Gamma^\pm} \in L^2(0,T;L^{4/3}(\omega)),\ h^{-1/4}\frac{d^\alpha \overset{r_i}{g}_i}{dt^\alpha} \in L^2(0,T;L^{4/3}(\gamma_1)),$$

$$h^{-1/6}\frac{d^\alpha \overset{r_\varphi}{f^\varepsilon}}{dt^\alpha} \in L^2(0,T;L^{6/5}(\omega)),\ \lambda_\pm^{3/4}\frac{d^\alpha}{dt^\alpha}g^\varphi\big|_{\Gamma^\pm} \in L^2(0,T;L^{4/3}(\omega)),\ h^{-1/4}\frac{d^\alpha \overset{r_\varphi}{g^\varphi}}{dt^\alpha} \in L^2(0,T;L^{4/3}(\gamma_1^\varphi)),$$

$$\lambda_\pm^{3/4}\frac{d^\alpha}{dt^\alpha}g^\psi\big|_{\Gamma^\pm} \in L^2(0,T;L^{4/3}(\omega)),\ h^{-1/4}\frac{d^\alpha \overset{r_\psi}{g^\psi}}{dt^\alpha} \in L^2(0,T;L^{4/3}(\gamma_1^\psi)),$$

$$h^{-1/6}\left(\overset{r}{f^\theta} - \frac{1-(-1)^{r+N_\theta}}{2}\overset{N_\theta+1}{f^\theta} - \frac{1+(-1)^{r+N_\theta}}{2}\overset{N_\theta+2}{f^\theta}\right) \in L^2(0,T;L^{6/5}(\omega)),$$

where $\alpha = 0,1$, $r_i = 0,...,N_i$, $i = 1,2,3$, $r_\varphi = 0,...,N_\varphi$, $r_\psi = 0,...,N_\psi$, $r_\theta = 0,...,N_\theta$, and $\vec{w}_{\mathbf{N}0} \in \vec{V}_{\mathbf{N}}(\omega)$, $\vec{w}_{\mathbf{N}1} \in \vec{H}_{\mathbf{N}}(\omega)$, $\zeta_{N_\theta 0} \in \vec{H}^\theta_{N_\theta}(\omega)$, then the problem (17)-(21) possesses a unique solution.

In the following theorem we present the results on the relationship between the obtained two-dimensional and the three-dimensional initial-boundary value problems, where we use the following spaces $H^{1,1,s}_{h^\pm}(\Omega) = \{v; \partial_3^{r-1} v \in H^1(\Omega), \partial_\alpha h^\pm \partial_3^r v \in L^2(\Omega), \alpha = 1,2, r = 1,...,s\}$, $s \in \mathbf{N}$.

Theorem 3. If $\Omega \subset \mathbf{R}^3$ is a bounded domain with Lipschitz boundary, $\tilde{\Gamma}_0^\varphi \neq \varnothing$, $\tilde{\Gamma}_0^\psi \neq \varnothing$, $\mathbf{u}_0 \in \mathbf{V}(\Omega)$, $\mathbf{u}_1 \in \mathbf{L}^2(\Omega)$, $\theta_0 \in L^2(\Omega)$, $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g}, \mathbf{g}' \in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, $f^\varepsilon, (f^\varepsilon)' \in L^2(0,T;L^{6/5}(\Omega))$, $g^\varphi, (g^\varphi)' \in L^2(0,T;\, L^{4/3}(\Gamma_1^\phi))$, $g^\psi, (g^\psi)' \in L^2(0,T;L^{4/3}(\Gamma_1^\psi))$, $f^\theta \in L^2(0,T;\, L^{6/5}(\Omega))$, $\rho, \chi \in L^\infty(\Omega)$, $\rho(x) > \alpha_\rho = const > 0$, $\chi(x) > \alpha_\chi = const > 0$, for almost all $x \in \Omega$, c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, η_{ij}, λ_{ij} $\in L^\infty(\Omega)$, $\partial \lambda_{ij} / \partial x_j \in L^3(\Omega)$, $i,j,p,q = 1,2,3$, conditions (10), (16) are fulfilled, and the functions $\mathbf{u}_{\mathbf{N}0} \in \mathbf{V}_{\mathbf{N}}(\Omega)$, $\mathbf{u}_{\mathbf{N}1} \in \mathbf{H}_{\mathbf{N}}(\Omega)$, $\theta_{N_\theta 0} \in H^\theta_{N_\theta}(\Omega)$, corresponding to the initial conditions $\vec{u}_{\mathbf{N}0} \in \vec{V}_{\mathbf{N}}(\omega)$, $\vec{u}_{\mathbf{N}1} \in \vec{H}_{\mathbf{N}}(\omega)$, $\vec{\theta}_{N_\theta 0} \in \vec{H}^\theta_{N_\theta}(\omega)$ of the two-dimensional problems, tend to $\mathbf{u}_0$, $\mathbf{u}_1$ and θ_0 in the spaces $\mathbf{H}^1(\Omega)$, $\mathbf{L}^2(\Omega)$ and $L^2(\Omega)$, respectively, as $N_{\min} = \min\limits_{1\le i\le 3}\{N_i, N_\varphi, N_\psi, N_\theta\} \to \infty$, then $\mathbf{u}_{\mathbf{N}}(t)$, $\varphi_{N_\varphi}(t)$, $\psi_{N_\psi}(t)$, $\theta_{N_\theta}(t)$ restored from the solutions $\vec{u}_{\mathbf{N}}$, $\vec{\varphi}_{N_\varphi}$, $\vec{\psi}_{N_\psi}$, $\vec{\theta}_{N_\theta}$ of the problem (17)-(21), possess the following properties

$$\begin{aligned} &\mathbf{u}_{\mathbf{N}}(t) \to \mathbf{u}(t) \quad \text{in } \mathbf{H}^1(\Omega), \\ &\mathbf{u}'_{\mathbf{N}}(t) \to \mathbf{u}'(t) \quad \text{in } \mathbf{L}^2(\Omega), \\ &\theta_{N_\theta}(t) \to \theta(t) \quad \text{in } L^2(\Omega), \end{aligned} \quad \begin{aligned} &\varphi_{N_\varphi}(t) \to \varphi(t) \quad \text{in } H^1(\Omega), \\ &\psi_{N_\psi}(t) \to \psi(t) \quad \text{in } H^1(\Omega), \end{aligned} \quad \text{for all } t \in [0,T], \text{ as } N_{\min} \to \infty.$$

In addition, if $d^r\mathbf{u}/dt^r \in L^2(0,T;\, (H^{1,1,s_r}_{h^\pm}(\Omega))^3)$, $s_r \in \mathbf{N}$, $r = 0,1,2$, $s_0 \ge s_1 \ge s_2 \ge 1$, $s_1 \ge 2$, $\varphi' \in L^2(0,T;\, H^{1,1,s_1^\varphi}_{h^\pm}(\Omega))$, $\psi' \in L^2(0,T;\, H^{1,1,s_1^\psi}_{h^\pm}(\Omega))$, $s_1^\varphi, s_1^\psi \in \mathbf{N}$, $s_1^\varphi \ge 2$, $s_1^\psi \ge 2$, $d^{\bar{r}}\theta / dt^{\bar{r}} \in L^2(0,T;$ $H^{1,1,s_{\bar{r}}^\theta}_{h^\pm}(\Omega))$, $s_{\bar{r}}^\theta \in \mathbf{N}$, $\bar{r} = 0,1$, $s_0^\theta \ge s_1^\theta \ge 1$, $s_0^\theta \ge 2$, then for suitable initial data $\vec{u}_{\mathbf{N}0}$, $\vec{u}_{\mathbf{N}1}$ and $\vec{\theta}_{N_\theta 0}$ the following estimate is valid

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_{\mathbf{N}}\|_{C([0,T];\mathbf{H}^1(\Omega))} + \|\mathbf{u}' - \mathbf{u}'_{\mathbf{N}}\|_{C([0,T];\mathbf{L}^2(\Omega))} + \|\varphi - \varphi_{N_\varphi}\|_{C([0,T];H^1(\Omega))} + \|\psi - \psi_{N_\psi}\|_{C([0,T];H^1(\Omega))} \\ &+ \|\theta - \theta_{N_\theta}\|_{C([0,T];L^2(\Omega))} + \|\theta - \theta_{N_\theta}\|_{L^2(0,T;H^1(\Omega))} \le \frac{1}{(N_{\min})^s} o(T, h^\pm, \mathbf{N}, N_\varphi, N_\psi, N_\theta), \end{aligned}$$

where $s = \min\{s_2, s_1 - 1, s_1^\varphi - 1, s_1^\psi - 1, s_1^\theta, s_0^\theta - 1\}$, $o(T, h^\pm, \mathbf{N}, N_\varphi, N_\psi, N_\theta) \to 0$, as $N_{\min} \to \infty$.

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