საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 12, №4, 2018
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 12, no.4, 2018

Mathematics

Extreme Points and Consistent Criteria for Hypothesis Testing in a Banach Space of Measures

Zurab Zerakidze* **and Mzevinar Patsatsia****

*Faculty of Mathematics , Gori State University, Gori, Georgia
**Department of Mathematics, Faculty of Mathematics and Computer Sciences, Sokhumi State University, Tbilisi, Georgia

(Presented by Academy Member Elizbar Nadaraya)

ABSTRACT. The consistent criteria for testing hypothesis are defined. It is shown that the probability of any kind of errors is zero for the specified criteria. The necessary and sufficient conditions for the existence of these criteria are considered. Also, the conditions for the existence of extreme points are proved. *© 2018 Bull. Georg. Natl. Acad. Sci.*

Key words: consistent criteria, singular, weakly separable, strongly separable

In a general theory of hypothesis testing there often arises a problem of passing from a weakly separable family of probability measures to the corresponding strongly separable family. Z. Zerakidze (see [1-4]) proved in terms of Zermelo-Fraenkel set theory that for a countable family of probability measures the notions of weak separability, separability, orthogonality and strong separability are equivalent.

Consistent Criteria for Hypothesis Testing

Let (E,S) be a measurable space with some given family of probability measures $\{\mu_i, i \in I\}$. We recall the following definitions from [5-9] which we need for our discussion.

Definition 2.1. An object $\{E, S, \mu_i, i \in I\}$ is called a statistical structure.

Definition 2.2. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal, if the probability measures $\{\mu_i, i \in I\}$ contained in it are pairwise singular measures.

Definition 2.3. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable, if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that the following relations are fulfilled:

$$\mu_i\left(X_j\right)=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$$

Definition 2.4. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly separable, if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that the following relations are fulfilled:

$$\mu_i(X_i)=1,\ \forall i\in I;$$
$$X_i\cap X_j=\varnothing,\ \forall i,j,\ i\neq j,\ i,j\in I;$$
$$\bigcup_{i\in I}X_i=E.$$

Remark 2.1. Strong separability implies weak separability, and weak separability implies orthogonality, but not vice versa.

Example 2.1. Let $E=[0,1]\times[0,1]$, S be the Borel σ-algebra of subsets of E. Take the S-measurable sets $X_i=\{x|\ 0\le x\le 1,\ y=i,\ i\in[0,1]\}$, and assume that ℓ_i are linear Lebesgue measures on X_i. Then the statistical structure $\{E,S,\ell_i,i\in[0,1]\}$ is strongly separable.

Example 2.2. Let $E=[0,1]\times[0,1]$, S be the Borel σ-algebra of subsets of E. Take the S-measurable sets $X_i=\{(x,y)|\ 0\le x\le 1,\ y=i,\ \text{if } i\in[0,1];\ x=i-2,\ 0\le y\le 1,\ \text{if } y\in[2,3]\}$.

Let ℓ_i be linear Lebesgue measures on X_i. Then the statistical structure $\{E,S,\ell_i,i\in[0,1]\cup[2,3]\}$ is weakly separable, but not strongly separable.

Example 2.3. Let $E=[0,1]\times[0,1]$, S be the Borel σ-algebra of subsets of E. Take the S-measurable sets $X_i=\{(x,y)|\ 0\le x\le 1,\ y=i,\ \text{if } i\in(0,1]\}$ and $X_i=E$ if $i=0$ and ℓ_0 are linear Lebesgue probability measures on $[0,1]\times[0,1]$. Then the statistical structure $\{E,S,\ell_i,i\in[0,1]\}$ is orthogonal, but not weakly separable.

Let H be the set of hypothesis and $\{\mu_h,h\in H\}$ be the probability measures defined on the measurable space (E,S). For each $h\in H$ denote by $\overline{\mu}_h$ the complement of the measure μ_h, and by $\text{dom}(\overline{\mu}_h)$ the σ-algebra of all measurable subsets $\overline{\mu}_h$ of E. Let

$$S_1=\bigcap_{h\in H}\text{dom}(\overline{\mu}_h).$$

Let H be the set of hypothesis and $B(H)$ be a σ-algebra of subsets of H which contains all finite subsets of H.

Definition 2.5. We say that the singular statistical structure $\{E,S_1,\overline{\mu}_h,h\in H\}$ admits a consistent criterion for hypothesis testing, if there exists at least one measurable mapping

$$\delta:(E,S_1)\to(H,B(H)),$$

such that

$$\overline{\mu}_h(\{x:\ \delta(x)=h\})=1\quad \forall h\in H.$$

Remark 2.2. The notion of a consistent criterion was introduced and studied by Z. Zerakidze (see [2]).

Definition 2.6. The probability

$$\alpha_h(\delta)=\overline{\mu}_h(\{x:\ \delta(x)\neq h\})$$

is called the probability of error of h-th kind for the given criterion δ.

Theorem 2.1. The statistical structure $\{E,S_1,\overline{\mu}_h,h\in H\}$ admits a consistent criterion δ for hypothesis testing, if and only if the probability of error of any kind is equal to zero for the criterion δ.

Proof. Necessity. Since the statistical structure $\{E,S_1,\overline{\mu}_h,h\in H\}$ admits a consistent criterion for hypothesis testing, there exists a measurable mapping

$$\delta:(E,S_1)\to(H,B(H)),$$

such that

$$\overline{\mu}_h\left(\left\{x:\ \delta(x)=h\right\}\right)=1 \quad \forall h\in H .$$

Therefore

$$\alpha_h(\delta)=\overline{\mu}_h\left(\left\{x:\ \delta(x)\neq h\right\}\right)=0 .$$

Sufficiency. Since the probability of error of any kind is equal to zero, we have

$$\alpha_h(\delta)=\overline{\mu}_h\left(\left\{x:\ \delta(x)\neq h\right\}\right)=0 \quad \forall h\in H .$$

On the other hand, $\left\{x\middle|\ \left(\delta(x)=h\right)\cup\delta(x)\neq h\right\}=E$ and

$$\overline{\mu}_h\left(\left\{x:\ \delta(x)=h\right\}\right)=1 \quad \forall h\in H .$$

Therefore δ is a consistent criterion for testing hypothesis. Theorem 2.1 is proved.

Consistent Criteria in Banach spaces

Definition 3.1. Let G be a σ -subalgebra of the σ -algebra S_1.

The algebra G is called free (with respect to a hypothesis $h\in H$), if all restrictions of the probability measures $\left\{\overline{\mu}_h,\, h\in H\right\}$ to the algebra G coincide.

Definition 3.2. A statistical structure $\left\{E,S_1,\overline{\mu}_h,\, h\in H\right\}$ is called separable, if the minimal σ -algebra D with respect to which all measurable functions of the form

$$h\to\overline{\mu}_h(A), \quad A\in S_1 ,$$

are measurable, separates the points on H .

Definition 3.3. A statistical structure $\left\{E,S_1,\overline{\mu}_h,\, h\in H\right\}$ is called strongly separable, if the σ -algebra D contains all finite subsets of H .

Definition 3.4. A statistical structure $\left\{E,S_1,\overline{\mu}_h,\, h\in H\right\}$ is called decomposable if there exist two subalgebras $S_2,S_3\subset S_1$, the union of which generates a σ -algebra S_1 . S_2 is sufficient and S_3 is free. The pair (S_2,S_3) is called the decomposition of a statistical structure $\left\{E,S_1,\overline{\mu}_h,\, h\in H\right\}$.

For any set $G\subset 2^H$ we denote by the symbol $\langle G\rangle$ the algebra generated by the set G, and by $\sigma\langle G\rangle$ the σ -algebra generated by the set G.

Let

$$I^*=\bigcap_{h\in H}\left\{A\in S_1:\ \mu_h(A)=0\right\}$$

Definition 3.5. An algebra $B_1\subset S_1$ is called minimal sufficient, if B_1 is sufficient, and for any sufficient algebra B_1' the condition $B_1\subset\sigma\left\langle B_1'\cup I^*\right\rangle$ is fulfilled.

Let $\Im$ be a σ -subalgebra of the algebra S_1 and μ be a probability measure defined on $\Im$. We denote by $S_\mu(S_1,\Im)$ the set of all finite and finitely additive extensions of the measure μ to the σ -algebra S_1. Let $ex\, S_\mu(S_1,\Im)$ be the set of its extreme points, $S_\mu^\sigma(S_1,\Im)$ be the set of all countably additive extensions of the measure μ to the σ -algebra S_1 and $ex\, S_\mu^\sigma(S_1,\Im)$ be the set of its extreme points.

As is known, $ex\, S_\mu(S_1,\Im)\neq\varnothing$, but the set $ex\, S_\mu^\sigma(S_1,\Im)$ may be empty ([7]).

Example 3.1. In the terminology of [8], let $ba(\Sigma,\nu,\Sigma')$ denote the set of all $\mu\in ba(S,\Sigma')$ with $\mu\geq 0$ and $\mu(S)=1$, such that $\frac{\mu}{\Sigma}=\nu$, where Σ' is an algebra of subsets of the set S , Σ is a subalgebra of the algebra Σ' and $\nu\in ba(S,\Sigma)$ with $\nu\geq 0$ and $\nu(S)=1$. Denote by $ca(\Sigma,\nu,\Sigma')$ the σ -sets, where ν is an arbitrarily defined probability measure on Σ, while Σ', $\Sigma\subset\Sigma'$, are σ -algebras. The set of all extreme points in $ca(\Sigma,\nu,\Sigma')$ is assumed to be empty. As an example, let us assume that S is the set of real numbers, $\Sigma=\{B|\ B\subset S\}$ (if B^c is countable), Σ' is the set of all Borel subsets of S, and ν is defined by $\nu(B)=0$ and $\nu(B)=0$ for countable B and B^c, respectively.

Let M^σ be a real linear space of all alternating finite measures on S.

Definition 3.6. A linear subset $M_H\subset M^\sigma$ is called a Banach space of measures, if:

The norm is defined on M_H so that M_H is a Banach space with respect to this norm, and the inequality

$$\|\mu+\lambda\nu\|\geq\|\mu\|$$

is fulfilled for any orthogonal measures $\mu,\nu\in M_H$ and real number $\lambda\neq 0$;

If $\mu\in M_H$ and $|f(x)|\leq 1$, then

$$\nu_f(A)=\int_A f(x)\nu(dx)\in M_B,$$

and $\|\nu_f\|\leq\|\nu\|$, where $f(x)$ is a measurable real function, $A\in S$;

If $\nu_n\in M_H$, $\nu_n>0$, $\nu_n(E)<+\infty$, $n=1,2,\dots$, and $\nu_n\downarrow 0$, then for any linear functional $\ell^*\in M_B^*$

$$\lim_{n\to\infty}\ell^*(\nu_n)=0.$$

Remark 3.1. The definition and construction of the Banach space of measures are given by Z. Zerakidze (see [4]). The proof of the next statement can also be found in [4].

Theorem 3.1. Let M_H be a Banach space of measures, then in M_H there exists a family of pairwise orthogonal probability measures $\{\mu_h,\, h\in H\}$ such that

$$M_H=\bigoplus_{h\in H}M_H(\overline{\mu}_h),$$

where $M_H(\overline{\mu}_h)$ is the Banach space of elements ν of the form

$$\nu(B)=\int_B f(x)\overline{\mu}_h(dx)\quad\forall B\in S,$$

such that

$$\int_E|f(x)|\overline{\mu}_h(dx)<+\infty$$

with the norm

$$\|\nu\|_{M_H(\overline{\mu}_h)}=\int_E|f(x)|\overline{\mu}_h(dx).$$

Theorem 3.2. Let $M_H=\bigoplus_{h\in H}M_H(\overline{\mu}_h)$ be the Banach space of measures, E be a completely separable metric space, S_1 be a Borel σ-algebra from E and card $H\leq 2^{\chi_0}$. Then, if the correspondence

$$f\leftrightarrow\ell_f,$$

given by the equality

$$\int_E f(x)\overline{\mu}_h(dx)=\ell_f(\overline{\mu}_h)\quad\forall\overline{\mu}_h\in M_H$$

is one-to-one, where ℓ_f is a linear continuous functional on M_H $(\ell_f\in M_H^*)$, $f\in F(M_H)$, where $F(M_H)$ is the set of all real functions f for which $\int_E f(x)\overline{\mu}_h(dx)$ is defined $\forall\overline{\mu}_h\in M_H$, Then the statistical structure $\{E, S_1, \bar{\mu}_h, h \in H\}$ admits a consistent criterion for hypothesis testing, and if the statistical structure $\{E, S_1, \bar{\mu}_h, h \in H\}$ is decomposable, then $\bar{\mu}_h \in ex\, S^{\sigma}_{\bar{\mu}_h}(S_1, S_3)\ \ \forall h \in H$.

Proof. For $f \in F(M_H)$ we define the linear continuous functional ℓ_f by the equality $\int_E f(x)\bar{\mu}_h(dx) = \ell_f(\bar{\mu}_h)$. Denote by I_f a countable subset of N, for which $\int_E f(x)\bar{\mu}_h(dx) = 0$ for $h \notin I_f$. Let to the functional ℓ_{f_h} on $M_H(\bar{\mu}_h)$ there correspond the functional $\ell_f(\bar{\mu}_h)$ on $M_B(\bar{\mu}_h)$. Then for $\bar{\mu}_{h_1}, \bar{\mu}_{h_2} \in M_B(\bar{\mu}_h)$ we have

$$\int_E f_{h_1}(x)\bar{\mu}_h(dx) = \ell_{f_{h_1}}(\mu_{h_2}) = \int_E f_1(x) f_2(x)\bar{\mu}_h(dx) = \int_E f_1(x)\bar{\mu}_h(dx).$$

Therefore $f_{h_1}(x) = f_1$ a.e. with respect to the measure $\bar{\mu}_h$. Let $f_{h_1}(x) > 0$ a.e. with respect to the measure $\bar{\mu}_h$ and

$$\int_E f_{h_1}(x)\bar{\mu}_{h_1}(dx) < \infty, \quad \tilde{\mu}_{h_1}(C) = \int_C f_{h_1}(x)\bar{\mu}_{h_1}(dx),$$

then

$$\int_E f_{h_1}(x)\bar{\mu}_{h_2}(dx) = 0 \quad \forall h \in H .$$

Hence it follows that $\bar{\mu}_{h_i}\left(C_{h_j}\right) = 0 \quad \forall j \neq i$. On the other hand, $\bar{\mu}_{h_i}\left(E - C_{h_j}\right) = 0$. Therefore the statistical structure $\{E, S_1, \bar{\mu}_h, h \in H\}$ is weakly separable and

$$\bar{\mu}_{h'}(C_{h''}) = \begin{cases} 1 \text{ if } h' = h'', \\ 0 \text{ if } h' \neq h''. \end{cases}$$

Let us write $\{\bar{\mu}_h, h \in H\}$, card $H \leq 2^{\chi_0}$, as an inductive sequence $\bar{\mu}_h < \omega_1$, where ω_1 denotes the first uncountable number of the power of the set H . Since the statistical structure $\{E, S_1, \bar{\mu}_h, h \in H\}$ is weakly separable, there exists a family of S_1 -measurable sets $\{X_h, h \in H\}$ such that the following relations are fulfilled:

$$\bar{\mu}_{h'}(X_{h''}) = \begin{cases} 1, \text{ if } h' = h'', \\ 0, \text{ if } h' \neq h'' \end{cases} \quad \forall h', h'' \in [0, \omega_1).$$

The sequence ω_1 of parts Z_h of the space E is defined so that the following conditions are fulfilled:

Z_h is a Borel subset in E for all $h < \omega_1$;

$$Z_h \subset X_h \text{ for all } h < \omega_1 ;$$

$$Z_{h'} \cap X_{h''} = \varnothing \text{ for all } h' < \omega_1,\ h'' < \omega_1,\ h' \neq h'' ;$$

Let $Z_{h_0} = X_{h_0}$, Assume that the particular sequence $\left\{Z_{h_j}\right\}_{j<i}$ has already been defined for $i < j$. As is known (see [3], $\mu^*\left(\bigcup_{j<i} Z_{h_j}\right) = 0$. Therefore there exists a Borel subset Y_{h_i} of the space E such that the following relations are valid: $\bigcup_{j<i} Z_{h_j} \subset Y_{h_i}$ and $\bar{\mu}\left(Y_{h_i}\right) = 0$. Assume that $Z_{h_i} = X_{h_i} - Y_{h_i}$, then we construct the sequence $\left\{Z_{h_i}\right\}_{i<\omega_1}$ which is a disjunctive measurable subset of the space E . Therefore $\bar{\mu}_h(X_h) = 1$

$\forall h < \omega_1$. Since the statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$, card $H \le 2^{\chi_0}$, is strongly separable, there exists a family $\{Z_h\}_{h\in H}$ of elements of the σ -algebra $S_1 = \bigcap_{h\in H} \operatorname{dom}(\overline{\mu}_h)$ such that:

$$\overline{\mu}_h(X_h) = 1 \ \ \forall h \in H;$$

$$Z_{h'} \cap Z_{h''} = \varnothing \ \ \forall h', h'' \in H, \ h' \ne h'';$$

$$\bigcup_{h\in H} X_h = E.$$

For $x \in E$, we put $\delta(x) = h$, where h is a unique hypothesis from the set H for which $x \in Z_{h'}$. The existence of such a unique hypothesis H is proved by using the conditions 2), 3). Now let $Y \in B(H)$. Then $\{x:\ \delta(x) \in Y\} = \bigcup_{h\in Y} Z_h$.

We have to show that $\{x:\ \delta(x) \in Y\} \in \operatorname{dom}(\overline{\mu}_{h_0})$ for each $h_0 \in H$. If $h_0 \in Y$, then

$$\{x:\ \delta(x) \in Y\} = \bigcup_{h\in H} Z_h = Z_{h_0} \cup \left(\bigcup_{h\in Y\setminus\{h_0\}} Z_h \right).$$

On the one hand, from the conditions 1), 2), 3) it follows that

$$Z_{h_0} \in S_1 = \bigcap_{h\in H} \operatorname{dom}(\overline{\mu}_h) \subseteq \operatorname{dom}(\overline{\mu}_{h_0})$$

On the other hand, the condition $\bigcup_{h\in Y\setminus\{h_0\}} Z_h \subseteq (E - Z_{h_0})$ implies that

$$\overline{\mu}_{h_0}\left(\bigcup_{h\in Y\setminus\{h_0\}} Z_h \right) = 0.$$

The last equality implies that $\bigcup_{h\in Y\setminus\{h_0\}} Z_h \in \operatorname{dom}(\overline{\mu}_{h_0})$. Since $\operatorname{dom}(\overline{\mu}_{h_0})$ is a σ -algebra, we deduce that

$$\{x:\ \delta(x) \in Y\} = Z_{h_0} \cup \left(\bigcup_{h\in Y\setminus\{h_0\}} Z_h \right) \in \operatorname{dom}(\overline{\mu}_{h_0}).$$

If $h_0 \notin Y$, then

$$\{x:\ \delta(x) \in Y\} = \bigcup_{h\in H} Z_h \subseteq (E - Z_{h_0})$$

and we conclude that $\overline{\mu}_{h_0}\{x:\ \delta(x) \in Y\} = 0$. The last equality implies that $\{x: \delta(x) \in Y\} \in \operatorname{dom}(\overline{\mu}_{h_0})$ for any $h_0 \in H$. Hence

$$\{x:\ \delta(x) \in Y\} \in \bigcap_{h\in H} \operatorname{dom}(\overline{\mu}_{h_0}) = S_1.$$

We have shown that the map $\delta : (E, S_1) \to (H, B(H))$ is measurable and $\overline{\mu}_h\{x:\ \delta(x) = h\} = 1$ $\forall h \in H$.. Thus the statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ admits a consistent criterion for testing hypothesis $\delta : (E, S_1) \to (H, B(H))$.

Let u be the linear operator defined by the formula

$$u(f) = \int_E f(x)\overline{\mu}_h(dx), \ \ f \in B(E, S_1).$$

The operator u is a positive isometric operator with norm $\|u\|=1$ and $u:B(E_1,S_1)\to(H,B(H))$ $\left(B(E_1,S_1)=(H,B(H))\right)$. The σ-algebra $\delta^{-1}(H)$ is a minimal sufficient σ-algebra. In the sequel $B(E,S_1)$ always denotes the vector space formed by all bounded real measurable functions on (E,S_1) having the natural order and norm

$$\|f\|=\sup_{x\in E}|f|.$$

Since

$$S_1=\bigcap_{h\in H}\operatorname{dom}(\overline{\mu}_h),$$

we have

$$S_1=\sigma\left\langle\delta^{-1}(B(H))\cup\Im^*\right\rangle,$$

where

$$\Im^*=\bigcap_{h\in H}\left\{A\in S:\ \mu_h(A)=0\right\}.$$

If the statistical structure $\{E,S_1,\overline{\mu}_h,h\in H\}$ is decomposable $\left(\delta^{-1}B(H),G_2\right)$, where $\delta^{-1}B(H)$ is a sufficient algebra and G_2 is a free algebra, then the algebras S_1 and G_2 are also decomposable and $\forall A\in S_1,\ A=C\Delta I$, $C\in\delta^{-1}(B(H))$, $I\in\Im^*$ and $\mu_h(A\Delta C)=0$. This means that $\mu_h\in\exp S_{\mu}^{\sigma}\left(S_1,\delta^{-1}(B(H))\right)$ $\forall h\in H$ (see [7, Theorem 1]).

მათემატიკა

ექსტრემალური წერტილები და ჰიპოთეზათა შემოწმების ძალდებული კრიტერიუმები ბანახის ზომათა სივრცეში

ზ. ზერაკიძე[*] და მ. ფაცაცია[**]

[*]გორის უნივერსიტეტი, მათემატიკის ფაკულტეტი, გორი, საქართველო
[**]სოხუმის სახელმწიფო უნივერსიტეტი, მათემატიკის და კომპიუტერულ მეცნიერებათა ფაკულტეტი, მათემატიკის დეპარტამენტი, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

ნაშრომში აგებულია ისეთი ძლიერად განცალებადი სტატისტიკური სტრუქტურა, რომლისთვისაც არსებობს ჰიპოთეზათა შემოწმების ძალდებული კრიტერიუმი. ამ კრიტერიუმის საშუალებით ყველა რიგის შეცდომის ალბათობა ნულის ტოლია. დამტკიცებულია აუცილებელი და საკმარისი პირობები ჰიპოთეზათა შემოწმების ძალდებული კრიტერიუმის არსებობის შესახებ.

REFERENCES

1. Zerakidze Z. (1984) On weakly divisible and divisible families of probability measures. *Bull. Acad. Sci. Georgia SSR* **113:** 273-275.
2. Zerakidze Z. (1991) Banach space of measure. Grigelionis B. et al. (eds), Probability Theory and Mathematical Statistics: *Proceedings of the Fifth Vilnius Conference*, **2**: 609-616. VSP/Mokslas Vilnius.
3. Eliauri L., Mumladze H., Zerakidze Z. (2013) Consistent criteria for checking hypothesis. *Jounal of Mathematics and System Science,* **3,** 10: 514-518.
4. Aleksidze L., Mumladze H., Zerakidze Z. (2014) The Consistent criteria for checking hypothesis. *Modern Stochastics: Theory and Applications* **1**, 1: 3-11.
5. Ibramhalilov I., Skorokhod A. (1980) Consistent criteria of parameters of random processes. Kiev.
6. Zerakidze Z. (2005) Generalization of Neumann-Pearson criterion. *Collection of Scientific Works,* pp. 63-69 *(in Georgian), Gori State University*.
7. Plachky L. (1976) Extremal and monogenic additive set functions. *Proceedings of the American Mathematical Society*, **54:** 193-196.
8. Dunford N., Schwatz J. T. (1958) Linear Operators. Part I: *General Theory*. Interscience Publishers, New York, London.
9. Zerakidze Z., Patsatsia M. (2017) The consistent estimators for homogeneous Gaussian fields of statistical structures. *IOSR Journal of Mathematics (IOSR-JM),* **13**, Issue 3: 01-06.

Received August, 2018