Informatics

The Probabilistic Model of Canonically Conjugate Fuzzy Subsets

Guram Tsertsvadze

N. Muskhelishvili Institute of Computational Mathematics, Georgian Technical University, Tbilisi, Georgia

(Presented by Academy Member Mindia Salukvadze)

ABSTRACT. The aim of this work consists in introduction of a new concept of canonically conjugate fuzzy subset containing a new information on an informational unit. It is known that an informational unit is a four (quadrupole - object, sing, value, verity). It is necessary to distinguish between the notions of inaccuracy and uncertainty. Inaccuracy is related with the “value” content of the four, whereas the uncertainty is related with the plausibility in the sense of its correspondence to reality (the component “verity”). It is known that there is a contradiction between the increase of the content of statement and its uncertainty saying that the increase of the expression’s accuracy increases its uncertainty, and vice versa, an uncertain character of information leads to some inaccuracy of the final conclusion received from the information in question. We see that on the one hand the notions are in a certain contradiction, and on another hand they complete each other. We have solved out the arising situation by use of a new concept of an optimal pair of canonically conjugate fuzzy subsets. Usually fuzzy subsets are being constructed based on an expert estimation of one of the concurrent components. A new approach to the representation of subjective and objective information based on the theory of fuzzy subsets is proposed. In order to consider collectively canonically conjugate attributes (inaccuracy and uncertainty), a probabilistic model of canonically conjugate subsets is constructed that essentially uses the theory of representations of functions of non-commutating variables. Within the framework of this model operator’s properties corresponding to the canonically conjugate attributes are established. © 2018 Bull. Georg. Natl. Acad. Sci.

Key words: fuzzy subset, membership function (compatibility), colour operator, uncertainty principle

One of the central concepts of our work is the concept of colour, attribute, each element of the universal set $\Omega$. Each element $\omega \in \Omega$ corresponds to a certain range of colour values (similar to the case in the case of real colour, for example, a certain frequency interval corresponds to the blue colour on the frequency scale).

The idea of introducing a new concept of colour as well as the most important concept of canonically conjugate fuzzy subsets belongs to T. Gachechiladze [1].

Here we use the well-known scheme for representing fuzzy subsets [2]. Let the set $\Omega$ (universal set) and the property $\mathcal{P}$ (attribute, colour) defined in it be given. We denote by $\mathcal{P}\{\Omega\}$ and $\mathcal{P}\{\Omega\}$ the subsets
$\Omega$ formed by the elements $\omega \in \Omega$ for which the sentence $P[\omega]$ has the color $P$ - true, or false. Further let $P_0 \subseteq P_{a1}$. We can consider the colour $\neg P$ defined in $\Omega$:

$$\neg P[\omega] \iff (\omega \in P_{a1}(\Omega)).$$

If $\neg P$ denotes a colour complementary to $P$, then $\Omega$ has the relation:

$$\neg P[\omega] \Rightarrow \neg P[\omega].$$

The inverse implication is valid only on the set $A(\Omega) = P_1(\Omega) \cup P_{a1}(\Omega)$. With the help of $P_0$, it is possible to define various $\neg P$, such that if $\neg P[\omega]$ is true, then $P[\omega]$ is false on $\Omega$, but the reverse implication takes place on $A(\Omega) \subseteq \Omega$. Let's see how it is possible to construct the set $P_0(\omega) \subseteq P_{a1}(\Omega)$.

To this end, we assume that each element of $\Omega$ can have a different colour $\neg P$ in different degrees. Further, suppose that we are able to assign to each $\omega \in \Omega$ a measure of its compatibility with the colour $P$. We formally define the mapping:

$$\mu_P : \Omega \to [0,1], P_0(\omega).$$

what

$$P[\omega] \iff (\mu_P(\omega) = 1).$$

For each $\omega \in \Omega$, $\mu_P(\omega)$ is called the value of the compatibility function $\omega$ with $P$. If $\mu_P(\omega) = 1$, we will say that $\omega$ has the colour $P$. If $\mu_P(\omega) = 0$, then $\omega$ does not have the colour $P$. In what follows, $P_0(\omega)$ is identified with a subset of elements $P_{a1}(\Omega)$ that do not possess the colour $P$. We call the colour $P$ in $\Omega$ satisfying the conditions considered above "measurable in $\Omega". If we assume in addition that $P_1(\Omega)$ is not empty, then $P$ will be called "completely measurable in $\Omega"."

**Assumption 1** (basic). The numerical characteristic of colour $\xi_P$ is a random variable. In the reference system of the universal set $\Omega$ this is a latent parameter.

Let the probability distribution of the values $\xi_P(x) \in R$ be characterized by the density $\rho_P(x_\omega)$,

$$\int_{R} \rho_P(x_\omega) dx_\omega = 1.$$

Quantity

$$x_\omega^* = M\xi_P = \int_{R} \rho_P(x_\omega) dx_\omega = 1,$$

mathematical expectation, we call the calculated value of the colour $P$ at the point $\omega$ of the universal set $\Omega$.

Note that formula (1) establishes the relation between the set of computable values and the universal set $\Omega$.

Therefore the meaning of the notations is clear:

$$\begin{cases}
\text{(the set } x_\omega^* \in R \text{ and } \omega \in P(\Omega) \Rightarrow P(R)), \\
\text{(the set } x_\omega^* \in R \text{ and } \omega \in P_1(\Omega) \Rightarrow P_1(\Omega)), \\
\text{(the set } x_\omega^* \in R \text{ and } \omega \in P_0(\Omega) \Rightarrow P_0(\Omega)), \\
\text{(the set } x_\omega^* \in R \text{ and } \omega \in P_{a1}(\Omega) \Rightarrow P_{a1}(\Omega)).
\end{cases}$$

We transferred the structure of the uncertainty from $\Omega$ to $R$.

If the set $P_1(\Omega)$ is not empty, then there exist $\omega$ such that

$$\int_{P(R)} \rho_P(x_\omega) dx_\omega \leq 1.$$
The presence of colour for \( \omega \in \Omega \), in addition to the quantity \( M_{\xi_\rho} \), is characterized by the variance \( \sigma^2_{\rho}(\omega) \):

\[
\sigma^2_{\rho}(\omega) = D(\xi_\rho) = \int_R (x_\omega - x_\omega^*)^2 \rho_\rho x_\omega dx_\omega . \tag{2}
\]

In the new model, we attribute \( \sigma^2_{\rho}(\omega) \) to the uncertainty of the colour value \( \mathcal{P} \) of \( \omega \). If \( \sigma^2_{\rho}(\omega) \to 0 \) then we say that \( \mathcal{P} \) in \( \omega \) has a well-defined value. The larger \( \sigma^2_{\rho}(\omega) \), the more uncertain \( \mathcal{P} \) is in \( \omega \). If \( \sigma^2_{\rho}(\omega) \to \infty \), then \( \omega \) does not have the colour \( \mathcal{P} \).

We denote the compatibility function (accessory) of a fuzzy subset by \( \mu_\rho(\omega) \). Thus, if \( \mu_\rho(\omega) = 1 \), then we say that \( x_\omega^* \) definitely has the colour \( \mathcal{P} \), and if \( \mu_\rho(\omega) = 0 \), we say that \( x_\omega^* \) does not have the colour \( \mathcal{P} \).

\( \mathcal{P}_0 \) is matched with the set of "unpainted" in colour \( \mathcal{P} \) elements \( x_\omega^* \in R \). Elements of \( R \) that do not belong to \( \mathcal{P}_0 \) (\( \mathcal{P}_1 \)) have a colour \( \mathcal{P} \) to some extent characterized by the number \( \mu_\rho(\omega) \in (0, 1) \). Otherwise, the colour model can be applied in \( R \). Below we will assume that the universal set \( \Omega \) is a numerical set \( \mu \mapsto \mu_\rho(x_\omega^*) \). The notion of measurability in \( \Omega \) corresponds to the notion of computability in \( R \), the notion of "complete measurability" in \( \Omega \) - "complete computability" in \( R \).

Below we will assume that the universal set is a numerical set \( R \):

\[ \mu_\rho (\omega) = \mu_\rho (x_\omega^*) . \]

**Assumption 2.** In \( R \) we define only \( \mathcal{P} \) and \( \neg \mathcal{P} \), that is, along with \( \mathcal{P}_1 \) (\( \mathcal{P}_0 \)) there is a unique \( \mathcal{P}_0 \) and the elements of \( R \) not belonging to these two subsets have an "intermediate" colour. This circumstance is expressed by means of the relation:

\[ \mu_{\neg \mathcal{P}} (x_\omega^*) = 1 - \mu_\mathcal{P} (x_\omega^*) . \]

**Definition 1.** For \( \forall \omega \in \Omega \) we introduce a certain interval of values \( \mathcal{P}(R) \mathcal{I}_\rho (\omega) \subseteq R \) with the help of the relation:

\[ \mu_\rho (\omega) = \mu_{\mathcal{I}_\rho (\omega)} (x_\omega^*) = \int_{I_{\mathcal{I}_\rho (\omega)}} \rho(x_\omega) dx_\omega = \int_{R} I_{\mathcal{I}_\rho (\omega)} (x_\omega) dx_\omega , \tag{3} \]

where \( I_{\mathcal{I}_\rho (\omega)} (x_\omega) \) is determined by the expert in such a way that \( \mu_{\mathcal{I}_\rho (\omega)} (\omega) \) is a function of compatibility \( \omega \) with colour \( \mathcal{P} \).

We call the interval defined by (3) the characteristic colour interval of \( \mathcal{P} \). We denote by \( \mathcal{I}_\rho \) the fuzzy subset corresponding to the colour \( \mathcal{P} \).

If \( \mu_{\mathcal{I}_\rho (\omega)} (\omega) \) is considered as the splitting function of the set (the interval \( \cup_{\omega \in R} I_{\mathcal{I}_\rho (\omega)} (x_\omega) = \text{Supp} \mathcal{I}_\rho \)), then for \( \forall \omega \in \Omega \) we can write

\[ I_{\mathcal{I}_\rho (\omega)} (x_\omega^*) = \mu_{\mathcal{I}_\rho (\omega)} (\omega) I_{\mathcal{I}_\rho (\omega)} (x_\omega^*) + (1 - \mu_{\mathcal{I}_\rho (\omega)} (\omega)) I_{\mathcal{I}_\rho (\omega)} (x_\omega^*) . \tag{4} \]

The first term \( \mu_{\mathcal{I}_\rho (\omega)} (\omega) I_{\mathcal{I}_\rho (\omega)} (x_\omega^*) \) is the value of the compatibility function of the fuzzy subset \( \Omega \), and the second term corresponds to the dual one:

\[ (1 - \mu_{\mathcal{I}_\rho (\omega)} (\omega)) I_{\mathcal{I}_\rho (\omega)} (x_\omega^*) = \mathcal{I}_\rho^D (\omega) . \]

**Definition 2.** The split set defined by the basic assumption 1, relations (1), (2) and (3), that is

\[ \Omega_{\mathcal{I}_\rho} = \{ \omega = (\omega, \mu_\rho(\omega)) : \omega \in \Omega \} \]

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will be called the probabilistic model of a fuzzy subset of the universal set \( \Omega \).

Similarly, a subset
\[
\tilde{\Omega}_p = \{ \tilde{\omega}_c = (\omega_c, \mu_p(\omega_c) : \omega_c \in \Omega \}
\]
will be called the probabilistic model of the fuzzy subset of the universal set \( \Omega^c \) that is canonically conjugate with respect to (4).

Both subsets \( \tilde{\Omega}_p \) and \( \tilde{\Omega}_p^c \) contain information about the state \( \omega \), which is complementary to each fuzzy subset \( \Omega_p \) and \( \tilde{\Omega}_p^c \).

Consider the operators
\[
\hat{M}(\alpha) = e^{ic\hat{P}}, \quad \hat{M}^c(\beta) = e^{ic\hat{P}^c},
\]
Scalar products
\[
\langle \alpha | M | x_{\omega}, \mathcal{P} \rangle = \langle \alpha | x_{\omega}, \mathcal{P}^c \rangle, \quad \langle \alpha | M^c | x_{\omega}, \mathcal{P} \rangle = \langle \alpha | x_{\omega}, \mathcal{P}^c \rangle,
\]
will be called the characteristic functions of the canonically conjugate colours \( \mathcal{P} \) and \( \mathcal{P}^c \), respectively
(Here we use the bracket notation Dirac [3]).

The following relations hold
\[
\langle x_{\omega}, \mathcal{P} | \hat{P} | x_{\omega}, \mathcal{P}^c \rangle = \langle x_{\omega}, \mathcal{P}^c | \hat{P} | x_{\omega}, \mathcal{P} \rangle, \quad \langle x_{\omega}, \mathcal{P} | \hat{P}^c | x_{\omega}, \mathcal{P}^c \rangle = \langle x_{\omega}, \mathcal{P}^c | \hat{P}^c | x_{\omega}, \mathcal{P} \rangle. \quad (5)
\]

**Theorem 1.** The operators of canonically conjugate colours \( \hat{P} \) and \( \hat{P}^c \) are connected by the following commutation relation
\[
\hat{P}\hat{P}^c - \hat{P}^c\hat{P} = ic\hat{E},
\]
where \( \hat{E} \) is the operator of the identity transformation.

**Proof.** Let \( \tilde{\mathcal{P}}(x) = x f(x), \ f(x) \) and \( f'(x) \in \tilde{L}(\mathbb{R}) \). Then
\[
(\tilde{\mathcal{P}}\tilde{\mathcal{P}}^c - \tilde{\mathcal{P}}^c\tilde{\mathcal{P}}) f(x) = \tilde{\mathcal{P}} (\tilde{\mathcal{P}} f(x) - \tilde{\mathcal{P}}^c (\tilde{\mathcal{P}} f(x)) = icE(x)f(x).
\]

Here we used the well-known formulas valid for Hermitian operators.

**Comment.** The relation (6) determines the quantitative relationship between canonically conjugate colours. This ratio limits the simultaneous "measurability" of canonically conjugate colours and allows us to study the uncertainty principle for "estimates" \( \mathcal{P} \) and \( \mathcal{P}^c \) simultaneously [4], analogous to the Heisenberg uncertainty principle.

It is obvious that the relations (5) can be generalized:
\[
\langle x_{\omega}, \mathcal{P} | \hat{P}^a | x_{\omega}, \mathcal{P}^c \rangle = \langle x_{\omega}, \mathcal{P}^c | \hat{P}^a | x_{\omega}, \mathcal{P} \rangle,
\]
Since
\[
e^{ia\hat{P}} = \sum_{k=0}^{\infty} \frac{(ai)^k}{k!} \hat{P}^k \quad \text{and} \quad e^{ib\hat{P}^c} = \sum_{k=0}^{\infty} \frac{(ai)^k}{k!} \hat{P}^c^k,
\]
Therefore we have
\[
M(\alpha) = \langle x_{\omega}, \mathcal{P} | \hat{M}(\alpha) | x_{\omega}, \mathcal{P} \rangle = \langle x_{\omega}, \mathcal{P}^c | \hat{M}(\alpha) | x_{\omega}, \mathcal{P} \rangle \quad \text{and} \quad M^c(\beta) = \langle x_{\omega}, \mathcal{P} | \hat{M}^c(\beta) | x_{\omega}, \mathcal{P} \rangle = \langle x_{\omega}, \mathcal{P}^c | \hat{M}^c(\beta) | x_{\omega}, \mathcal{P} \rangle. \quad (7)
\]
In view of the fact that
\[ \hat{p} | x_{\alpha}; \mathcal{P} \rangle = x_{\alpha} | x_{\alpha}; \mathcal{P} \rangle, \quad \hat{p}^c | x_{\alpha^c}; \mathcal{P}^c \rangle = x_{\alpha} | x_{\alpha^c}; \mathcal{P}^c \rangle, \]
and for the operators \( \hat{e} \) and \( \hat{e}^c \) fair the relations
\[ \hat{e} | x_{\alpha}; \mathcal{P} \rangle = ic \frac{d}{dx_{\alpha}} | x_{\alpha}; \mathcal{P} \rangle, \quad \hat{e}^c | x_{\alpha}; \mathcal{P} \rangle = -ic \frac{d}{dx_{\alpha}} | x_{\alpha}; \mathcal{P} \rangle. \]

According to (8), (9), we can write:
\[ \hat{M} (\alpha) | x_{\alpha}; \mathcal{P} \rangle = e^{i\alpha x_{\alpha}} | x_{\alpha}; \mathcal{P} \rangle, \]
\[ \hat{M} (\alpha) | x_{\alpha^c}; \mathcal{P}^c \rangle = x_{\alpha} | x_{\alpha^c}; \mathcal{P}^c \rangle, \]
\[ \hat{M} (\beta) | x_{\alpha}; \mathcal{P} \rangle = x_{\alpha} | x_{\alpha^c}; \mathcal{P} \rangle, \]
\[ \hat{M} (\beta) | x_{\alpha^c}; \mathcal{P}^c \rangle = e^{i\beta x_{\alpha}} | x_{\alpha^c}; \mathcal{P}^c \rangle. \]

The first and last equalities in (10) are directly expressed from (8), as regards the second and third equalities, then, according to (9), we have:
\[ \sum_{k=0}^{\infty} \frac{(\alpha i)^k}{k!} \hat{p}^k | x_{\alpha}; \mathcal{P} \rangle = \sum_{k=0}^{\infty} \frac{(-\alpha c)^k}{k!} \frac{d^k}{dx_{\alpha}^k} | x_{\alpha^c}; \mathcal{P}^c \rangle = x_{\alpha} | x_{\alpha^c}; \mathcal{P}^c \rangle, \]
\[ \sum_{k=0}^{\infty} \frac{(\beta i)^k}{k!} \hat{p}^k | x_{\alpha}; \mathcal{P} \rangle = \sum_{k=0}^{\infty} \frac{(\beta i)^k}{k!} \frac{d^k}{dx_{\alpha}^k} | x_{\alpha^c}; \mathcal{P}^c \rangle = x_{\alpha} | x_{\alpha^c}; \mathcal{P}^c \rangle. \]

**Theorem 2.** If \( \hat{M} (\alpha) \) and \( \hat{M} (\beta) \) are defined by formulas (10),
\[ | x_{\alpha}; \mathcal{P} \rangle \in L^2 (R) \text{ and } | x_{\alpha^c}; \mathcal{P}^c \rangle = \hat{F} | x_{\alpha}; \mathcal{P} \rangle, \]
that
\[ \rho_p (x_{\alpha}) = \frac{1}{2\pi} \int_{R} | \langle x_{\alpha^c}; \mathcal{P} | e^{i\alpha x_{\alpha}} | x_{\alpha}; \mathcal{P} \rangle e^{-i\alpha x_{\alpha}} | d\alpha, \]
\[ \rho_{p^c} (x_{\alpha^c}) = \frac{1}{2\pi} \int_{R} | \langle x_{\alpha}; \mathcal{P} | e^{i\beta x_{\alpha}} | x_{\alpha^c}; \mathcal{P} \rangle e^{-i\beta x_{\alpha}} | d\beta. \]

**Proof.** We now prove (11). Formula (12) is proved similarly. We have:
\[ \frac{1}{2\pi} \int_{R} d\alpha e^{-i\alpha x_{\alpha}} \langle x_{\alpha^c}; \mathcal{P} | e^{i\alpha x_{\alpha}} | x_{\alpha}; \mathcal{P} \rangle dx_{\alpha^c} = \]
\[ = \int dx_{\alpha^c} \langle x_{\alpha^c}; \mathcal{P} | x_{\alpha}; \mathcal{P} \rangle \left( \frac{1}{2\pi} \int_{R} d\alpha e^{i\alpha x_{\alpha}} \right) = \]
\[ = \int dx_{\alpha^c} \langle x_{\alpha^c}; \mathcal{P} | x_{\alpha}; \mathcal{P} \rangle \delta (x_{\alpha^c} - x_{\alpha}) = | x_{\alpha}; \mathcal{P} \rangle \hat{p} = \rho_p (x_{\alpha}). \]

In conclusion, we note that, by virtue of (7), the probability density of the canonically conjugate colours \( \mathcal{P} \) and \( \mathcal{P}^c \) admits the following representation:
\[ \rho_p (x_{\alpha^c}) = \frac{1}{2\pi} \int_{R} | \langle x_{\alpha^c}; \mathcal{P}^c | \hat{M} (\alpha) | x_{\alpha^c}; \mathcal{P}^c \rangle e^{-i\alpha x_{\alpha^c}} | d\alpha, \]
\[ \rho_{p^c} (x_{\alpha}) = \frac{1}{2\pi} \int_{R} | \langle x_{\alpha}; \mathcal{P} | \hat{M} (\beta) | x_{\alpha}; \mathcal{P} \rangle e^{-i\beta x_{\alpha}} | d\beta. \]
ნინო ჩხერშიძე

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1. Guram Tsertsvadze

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Received August, 2018