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Mathematics

On Absolute Continuity of Random Measures for an Integral Type Transformation

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ABSTRACT. In this paper random measures and their nonlinear transformations in an infinite dimensional linear space are considered. The conditions of absolute continuity for this measures are obtained in case of cylindrical type transformation of a space. Explicit formula for Radon-Nikodym derivative is given. *© 2019 Bull. Georg. Natl. Acad. Sci.*

Key words: equivalence of measures, random measures, nonlinear transformation

Random measures are of great practical importance because they surely appear as solutions of differential equations in measures (see [1, 2]), with random coefficients and additive noise.

In the present paper we study the absolutely continuity of distributions of random measures in the case of their nonlinear transformation. The Radon-Nykodim density is calculated, the density formula containing an extended stochastic integral over a random measure. The terms and notation used here are the same as in [3].

Let $(\Omega, \Im, \mathrm{P})$ be a fixed probability space. Let $\mu(\omega, A)$ be a real, random a.s. σ-additive function of sets on some measurable space $\{X, B(X)\}$. Assume that $E\mu(A)=0$ and there exists an σ-additive measure $\beta(A_1\times A_2)=E\mu(A_1)\mu(A_2)$ on $\{X\times X, B(X)\times B(X)\}$. Let H be the space of measurable function φ on $\{X, B(X)\}$ with real values and scalar product

$$(\phi,\psi)=\iint_{XX}\phi(x)\psi(y)\beta(dx\times dy)=E\int_X\phi(x)\mu(dx)\int_X\psi(y)\mu(dy)$$

Following [2], we will construct the conjugate space to H and realize it as a space of measures $\mu=\mu_\varphi=S_\varphi$ where S is a unitary operator

$$S: H\to H^*$$

and

$$\mu_\varphi(A) = \int_X \varphi(x)\beta(dx \times A),$$

so that the pairing of elements from H and H^* can be written in the form

$$\langle \psi, \mu_\phi \rangle = \langle \psi, S\phi \rangle = (\psi, \phi).$$

It is obvious that H^* is also a Hilbert space with the scalar product

$$(\nu_\phi, \nu_\psi)_* = (\phi, \psi).$$

As follows from [1], we can construct the embedding operator Q and the Hilbert space H_+ densely embedded in H and being such that Q is the Hilbert-Shmidt operator. Thus, we can construct a triple of Hilbert equipment

$$H_+ \overset{Q}{\subset} H \overset{S}{=} H^* \overset{\hat{Q}}{\subset} H_-$$

observing that the distribution $\widetilde{\mu}(\Delta) = \widetilde{\mu}_A(\Delta) \overset{def}{=} P\{\nu(A) \in \Delta\}$ is concentrated in H_-. Let $L(M, N)$ denote the space of real functionals defined on M and differentiated along constant elements $N \subset M$. In the sequel, it will be assumed that the distribution $\widetilde{\mu}$ is a smooth measure on H_- (see[1]). This means that there exists a measurable function $\lambda : H_- \to H_-$, called the logarithmic derivative of the measure $\widetilde{\mu}$ and being such that the formula of integration by parts

$$\int_{H_-} \langle \varphi, f'(\mu) \rangle \widetilde{\mu}(d\mu) = -\int_{H_-} f(\mu) \langle \varphi, \lambda(\mu) \rangle \widetilde{\mu}(d\mu)$$

holds for functions $f \in L(H_-, H_+)$. Recall that the expression $\rho_{\widetilde{\mu}}(\varphi, \mu) = < \lambda(\mu), \varphi > = \int_X \varphi(x)\lambda(\mu)(dx)$ is called a stochastic integral.

Theorem 1. **[3]** *Let $H_+ \subset H \subset H_-$ be an equipped Hilbert space of random measures, $\widetilde{\mu}$ is measure on H_- with the logarithmic derivative along constant directions from H_+ such as $\rho_{\widetilde{\mu}}(\varphi, \mu) = < \lambda(\mu), \varphi >$. Let $f : H_- \to H_-$ be an invertible transformation, the inverse of which is given by the formula*

$$f^{(-1)} : \mu \to \nu = \mu + F(\mu)$$

and the following conditions are met:

1) $F : H_- \to H_+$ is a continuously differentiable mapping;

2) The inverse of the linear operator $I + tF'(\mu)$ is bounded for $0 \le t \le 1,\ \mu \in H$.

Then the measures $\widetilde{\mu}$ and $\widetilde{\widetilde{\mu}} = \widetilde{\mu} f^{(-1)}$ are equivalent and the Radon-Nykodim derivative has the form

$$\frac{d\tilde{\tilde{\mu}}}{d\tilde{\mu}}(\mu) = \det(I + F'(\mu)) \exp\left\{\left\langle \int_0^1 \lambda(\mu + tF(\mu))\,dt, F(\mu) \right\rangle\right\}.$$

Using this theorem we study the problem of the equivalence of distributions of random values for a nonlinear transformation that has an integral form and depends on fixed elements of the original σ-algebra.

Let Q_m be an m -dimensional cube

$$Q_m = \prod_{k=1}^{m} [0,1] = [0,1]^m,$$

ν_m a measure on Borel subsets of Q_m. Let further $\{X, \mathrm{B}\}$ be some measurable space.

We will consider the random measures $\mu(t,A) = \mu(t,A,\omega)$ defined on $Q_m \times \mathrm{B} \times \Omega$ with real values, which have the following properties:

M1) μ is a random value for fixed $t \in Q_m$ and $A \in \mathrm{B}$; $E\mu^2(t,A) < \infty$ almost everywhere on the Lebesgue measure for all $t \in Q_m$ and $A \in \mathrm{B}$

M2) for fixed $t \in Q_m$, $\mu(t,A)$ is a measure with alternating signs almost everywhere for all $\omega \in \Omega$

M3) for fixed $A \in \mathrm{B}$, μ is square integrable with respect to a measure ν_m with probability 1.

Let $\tilde{L}_2(Q_m)$ be a space of random measures $\mu(t,A)$ with properties M1), M2) and M3). It is understood that $\tilde{L}_2(Q_m)$ is a Hilbert space with the scalar product

$$(\mu_1, \mu_2)_{\tilde{L}_2(Q_m)} = E \int_{Q_m} \mu_1(t,X)\mu_2(t,X)\nu_m(dt)$$

and, accordingly, with the norm

$$\|\mu\|^2_{\tilde{L}_2(Q_m)} = E \int_{Q_m} \mu^2(t,X)\nu_m(dt).$$

Denote the self-correlation measure by

$$\beta_{ts}^{\mu_1\mu_2}(A,B) = E\mu_1(t,A)\mu_2(s,B)$$

In terms of this measure we write

$$(\mu_1, \mu_2)_{\tilde{L}_2(Q_m)} = \int_{Q_m} \beta_{tt}^{\mu_1\mu_2}(X,X)\nu_m(dt)$$

and

$$\|\mu\|^2_{\tilde{L}_2(Q_m)} = \int_{Q_m} \beta_{tt}^{\mu\mu}(X,X)\nu_m(dt)$$

Let $\tilde{L}_2^+(Q_m) \subset \tilde{L}_2(Q_m) \subset \tilde{L}_2^-(Q_m)$ be a quasi-kernel equipment of the basic space $\tilde{L}_2(Q_m)$. Consider the transformation $\tilde{L}_2(Q_m)$

$$\tilde{\mu}(A,t) = \mu(A,t) + \int_{Q_m} G(t,s,A,\mu(A_1,s),\mu(A_2,s),\ldots,\mu(A_n,s))\nu_m(ds), \tag{1}$$

where $A_1, A_2, \ldots, A_n$ are fixed measurable sets from Band $G(t,s,A,x_1,x_2,\ldots,x_n)$ is a function on $Q_{2m} \times \mathrm{B} \times R^n$. We call such a mapping a cylindrical type transformation (see [2, 3]).

Let us introduce the notation

$$\int_{Q_m} G(t,s,A,\mu(A_1,s),\mu(A_2,s),\ldots,\mu(A_n,s))\nu_m(ds) = g(t,A,\mu_1,\mu_2,\ldots,\mu_n)$$

$$\left.\frac{\partial g(t, A_i, \mu_1, \ldots, \mu_{j-1}, u_j, \mu_{j+1}, \ldots, \mu_n)}{\partial u_j}\right|_{u_j=\mu_j} = g'_{ij}(t, A_i, \mu_1, \ldots, \mu_n)$$

The distribution of a random measure μ is denoted by P_μ. Transformation (1) of the space $\widetilde{L}_2(Q_m)$ changes the measure μ to the measure $\widetilde{\mu}$ the distribution of which is denoted by $P_{\widetilde{\mu}}$. According to the Minlos-Sazonov theorem, in the above-mentioned conditions these distributions are concentrated in the space $\widetilde{L}_2^-(Q_m)$. We are interested in the conditions for which these distributions are equivalent.

Theorem 2. *Let conditions M1), M2) and M3) be fulfilled for transformation (1),* $A_1, A_2, \ldots, A_n$ *be fixed measurable sets from B and the following conditions be fulfilled for function* $G(t, s, A, x_1, x_2, \ldots, x_n)$ *on* $Q_{2m} \times \mathrm{B} \times R^n$.

G1) for any $A \in \mathrm{B}$, $G(t, s, A, x_1, x_2, \ldots, x_n)$ *is continuous with respect to* $t, s \in Q_m$,*differentiable with respect to* $x_1, x_2, \ldots, x_n \in R$ *and square-integrable with respect to the measure* $\nu_m \times \nu_m \times l_n$, *where* l_n *is a Lebesgue measure in* R^n.

G2) for fixed $t, s \in Q_m$ *and* $x_1, x_2, \ldots, x_n \in R$, $G(t, s, A, x_1, x_2, \ldots, x_n)$ *is a measure on B.*

G3) there exists a nonzero determinant

$$\Delta(x) = \begin{vmatrix} 1+g'_{11}(x) & g'_{12}(x) & \ldots & g'_{1n}(x) \\ g'_{21}(x) & 1+g'_{22}(x) & \ldots & g'_{2n}(x) \\ \ldots & \ldots & \ldots & \ldots \\ g'_{n1}(x) & g'_{n2}(x) & \ldots & 1+g'_{nn}(x) \end{vmatrix}.$$

Then, if the covariation measure $\beta_{ts}^{\mu\nu}$ *has a logarithmic derivative with respect to each argument along the constant direction* $\widetilde{L}_2^+(Q_m)$, *then the measures* P_μ *and* $P_{\widetilde{\mu}}$ *are equivalent and*

$$\frac{dP_{\widetilde{\mu}}}{dP_\mu}(\mu) = \Delta(\mu)\exp\left\{-\beta(t, g(t, A, \mu), \mu) - \frac{1}{2}\left\|g(t, A, \mu)\right\|^2_{\widetilde{L}_2(Q_m)}\right\}, \tag{2}$$

where $\beta(t, g, A)$ *denotes some measurable functional which is an abstract analogue of an extended stochastic integral* ([2]).

Proof. For transformation we apply theorem 1. The main requirement in this theorem consists in proving that transformation (1) is invertible. To establish this fact, we substitute step-by-step A_1, A_2, ..., A_n in (1) and form the system

$$\widetilde{\mu}(t, A_1) = \mu(t, A_1) + \int_{Q_m} G(t, s, A_1, \mu(s, A_1), \ldots, \mu(s, A_n))\nu_m(ds)$$

$$\widetilde{\mu}(t, A_2) = \mu(t, A_2) + \int_{Q_m} G(t, s, A_2, \mu(s, A_1), \ldots, \mu(s, A_n))\nu_m(ds)$$

....

$$\widetilde{\mu}(t, A_n) = \mu(t, A_n) + \int_{Q_m} G(t, s, A_n, \mu(s, A_1), \ldots, \mu(s, A_n))\nu_m(ds).$$

According to condition G3) of the theorem, this system is solvable with respect to $\mu(t, A_1)$, $\mu(t, A_2)$, ..., $\mu(t, A_n)$. Therefore there exist functions H_1, H_2,..., H_n such that

$$\mu(t, A_1) = H_1(t, A_1, ..., A_n, \widetilde{\mu}(t, A_1), ..., \widetilde{\mu}(t, A_n))$$
$$\mu(t, A_2) = H_2(t, A_1, ..., A_n, \widetilde{\mu}(t, A_1), ..., \widetilde{\mu}(t, A_n))$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\mu(t, A_n) = H_n(t, A_1, ..., A_n, \widetilde{\mu}(t, A_1), ..., \widetilde{\mu}(t, A_n)).$$

Then from (1) we obtain an inverse transformation of the form

$$\mu(t, A) = \widetilde{\mu}(t, A) -$$
$$- \int_{Q_m} G(t, s, A, H_1(s, A_1, ..., A_n, \widetilde{\mu}(s, A_1), ..., \widetilde{\mu}(s, A_n)), ..., H_n(s, A_1, ..., \widetilde{\mu}(s, A_1), ..., \widetilde{\mu}(s, A_n)))\nu_m(ds).$$

We observe now that the smoothness (in a sense of the existence of a logarithmic derivative along the constant directions of the subspace $\widetilde{L}_2^+(Q_m)$) of a self-correlation measure gives the same smoothness of the distribution $P_{\widetilde{\mu}}$. The other conditions of the theorem are fulfilled automatically, which shows that the assertion we wanted to prove and formula (2) are valid.

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შემთხვევითი ზომების აბსოლუტურად უწყვეტობის შესახებ ინტეგრალური ტიპის გარდაქმნისას

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