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Mathematics

The Complete Regularity of some T_0 Topological Protomodular Algebras

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ABSTRACT. The paper deals with the problem whether it is possible to generalize one of the classical results of topological algebra asserting that any T_0 topological group is completely regular to the case, where ”group” is replaced by an algebra from a protomodular variety of universal algebras. The subvariety of such a variety is found for algebras of which the answer to this question is positive.

Key words: topological protomodular algebra; separation axioms; completely regular space; uniformity.

In [1] D. Bourn introduced the notion of a protomodular category as an abstract setting, in which many properties of groups remain valid. Later D. Bourn and G. Janelidze found the purely syntactical characterization of protomodular varieties of universal algebras [2]. Applying this characterization F. Borceux and M. M. Clementino generalized many properties of topological groups to the case of topological protomodular algebras [3-4]. Among them there was the implication $T_1 \Rightarrow T_3$. In view of the classical result of topological algebra asserting that any T_0 group is not only regular, but also completely regular the question arises whether one can replace T_3 by $T_{3\frac{1}{2}}$ in the above-mentioned implication for topological protomodular algebras. In the present paper we give the positive answer to this question in the case of the so-called right-cancellable protomodular algebras.

Let $\mathbb{C}$ be a category with finite limits, and let C be an object of $\mathbb{C}$. Let $Split(C)$ be the category of triples (A,i,s) with $i: C \to A$ and $s: A \to C$ such that $si = 1_C$; a morphism $(A,i,s) \to (A',i',s')$ in $Split(C)$ is a morphism $h: A \to A'$ of $\mathbb{C}$ such that $hi = i'$ and $s'h = s$.

Any morphism $f: C \to D$ induces by pullback the so-called inverse image functor

$$f^*: Split(D) \to Split(C).$$

$\mathbb{C}$ is called protomodular if the functor f^* reflects isomorphisms, for any f [1]. Note that in the case of a category $\mathbb{C}$ with a zero object, this condition is equivalent to the ”split” version of the classical ”short five lemma” [4].

Theorem 1. [2]. *Let* $\mathbb{V}$ *be a variety of universal algebras.* $\mathbb{V}$ *is protomodular if and only if its algebraic theory contains, for some natural n, constants* $e_1, e_2, ..., e_n$, *binary operations* $\alpha_1, \alpha_2, ..., \alpha_n$, *and an* $(n+1)$ *– ary operation* θ *such that the following identities are satisfied:*

$$\alpha_i(a,a) = e_i; \tag{1}$$

$$\theta\big(\alpha_1(a,b), \alpha_2(a,b), ..., \alpha_n(a,b), b\big) = a. \tag{2}$$

The operation θ satisfying (2) for some α_i and e_i which in their turn satisfy (1) is called protomodular.

The examples of protomodular varieties are given by the varieties of groups (more generally, varieties the algebraic theories of which contain group operations), left/right semi-loops, loops, locally Boolean distributive lattices [3], Heyting algebras and Heyting semi-lattices [5]. The algebraic theory of a protomodular variety might have more than one protomodular operation. For instance, as is well-known, the algebraic theory of the variety of Boolean algebras has a group operation. Another protomodular operation of this algebraic theory is given in [4]. The identities (1) and (2) immediately imply [4]:

Lemma 2. *Let* $\mathbb{V}$ *be a protomodular variety of universal algebras. Let* A *be a* $\mathbb{V}$*-algebra*, *and* $a, b, c \in A$.

(a) If $\alpha_i(a,c) = \alpha_i(b,c)$, *for any* i $(1 \le i \le n)$, *then* $a = b$;

(b) if $\alpha_i(a,b) = e_i$, *for any* i $(1 \le i \le n)$, *then* $a = b$;

(c) $\theta\big(e_1, e_2, ..., e_n, a\big) = a.$

One can prove

Lemma 3. *Let* $\mathbb{V}$ *be a protomodular variety, and let A be a* $\mathbb{V}$*-algebra. The conditions (i)-(iv) given below are equivalent:*

(i) for any $a_1, a_2, ..., a_n, a'_1, a'_2, ..., a'_n, b, b' \in A$ *and* i $(1 \le i \le n)$, *we have*

$$\alpha_i(\theta(a_1, a_2, ..., a_n, b), \theta(a'_1, a'_2, ..., a'_n, b)) =$$
$$= \alpha_i(\theta(a_1, a_2, ..., a_n, b'), \theta(a'_1, a'_2, ..., a'_n, b'));$$

(ii) for any $a_1, a_2, ..., a_n, a'_1, a'_2, ..., a'_n, b, b' \in A$ *and* i $(1 \le i \le n)$, *we have*

$$\alpha_i(\theta(a_1, a_2, ..., a_n, \theta(a'_1, a'_2, ..., a'_n, b)), \theta(a''_1, a''_2, ..., a''_n, b)) =$$
$$= \alpha_i(\theta(a_1, a_2, ..., a_n, \theta(a'_1, a'_2, ..., a'_n, b')), \theta(a''_1, a''_2, ..., a''_n, b'));$$

(iii) for any $a_1, a_2, ..., a_n, a'_1, a'_2, ..., a'_n, b, b' \in A$ *and* i $(1 \le i \le n)$, *we have*

$$\alpha_i(\theta(a_1, a_2, ..., a_n, b), \theta(a'_1, a'_2, ..., a'_n, \theta(a''_1, a''_2, ..., a''_n, b)) =$$
$$= \alpha_i(\theta(a_1, a_2, ..., a_n, b'), \theta(a'_1, a'_2, ..., a'_n, \theta(a''_1, a''_2, ..., a''_n, b'));$$

(iii) for any i $(1 \le i \le n)$, *there is a term* t_i *of 3n varriebles over the signature of* $\mathbb{V}$, *such that, for any* $a_1, a_2, ..., a_n, a'_1, a'_2, ..., a'_n, a''_1, a''_2, ..., a''_n, b, b' \in A,$ if

$$\theta(a_1, a_2, ..., a_n, b) = \theta(a'_1, a'_2, ..., a'_n, b'),$$

then

$$\alpha_i(\theta(a''_1, a''_2, ..., a''_n, b'), b) = t_i(a_1, a_2, ..., a_n, a'_1, a'_2, ..., a'_n, a''_1, a''_2, ..., a''_n).$$

For simplicity, algebras from a protomodular variety are called protomodular algebras.

Definition 4. *A protomodular algebra is called right-cancellable if it satisfies the equivalent conditions of Lemma 3.*

Example 5. Let $\mathbb{V}_2$ denote the simplest protomodular variety, i.e. the variety with the signature $\mathfrak{J}_2$ containing only one ternary operation symbol θ, the binary operation symbols α_1, α_2, and the constant symbols e_1, e_2, where the identities are (1) and (2) for $n = 2$.

Let $A = \{0,1\}$. Let us introduce the structure of $\mathbb{V}_2$-algebra on A as follows. Let $\theta(i, j, k) = k$ if $i \neq j$, and $\theta(i, j, k) = 1 - k$ if $i = j$. Moreover, let $\alpha_1(i, j)$ be 0, for any i, j; let $\alpha_2(i, j)$ be 0 if $i \neq j$, and be 1 if $i = j$. Besides, let $e_1 = 0$ and $e_2 = 1$. One can verify that A is a right-cancellable $\mathbb{V}_2$-algebra. Since the subcategory of right-cancellable protomodular $\mathbb{V}_2$-algebras is a subvariety of $\mathbb{V}_2$, the algebra $\left(\prod_{i\in I} A_i\right)/R$ is a right-cancellable $\mathbb{V}_2$-algebra, for any set I, $A_i = A$ $(i \in I)$, and any congruence R on $\prod_{i\in I} A_i$.

Remark 6. One can show that any right-cancellable left semi-loop (loop) is a group.

Remark 7. One can show that a non-trivial Boolean algebra and a non-trivial locally Boolean distributive lattice with respect to the protomodular operations given in resp. [4] and [3] are not right-cancellable. Similarly, a non-trivial Heyting algebra and a non-trivial Heyting semi-lattice with respect to the protomodular operations given in [5] are not right-cancellable.

Let now $\mathbb{V}$ be any variety of universal algebras, and let A be an algebra from $\mathbb{V}$. A is called a topological algebra if A is equipped with a topology such that all operations from the signature of $\mathbb{V}$ are continuous.

Proposition 8. [4]. *Let a be an element of a topological protomodular algebra A. Then the subsets*

$$\bigcap_{i=1}^{n} \alpha_i(-, a)^{-1}(H_i),$$

with H_i being an open neighborhood of e_i constitute a base of neighborhoods of a.

Lemma 9. *For any topological $\mathbb{V}$-algebra, the separation axiom T_0 implies T_1*.

Recall [6] that a uniformity on a set X is a set $\mathcal{U}$ of reflexive symmetric binary relations such that the following conditions are satisfied:

(i) if $R \in \mathcal{U}$ and $R \subseteq R'$, for a reflexive symmetric binary relation R', then $R' \in \mathcal{U}$;

(ii) if $R, R' \in \mathcal{U}$, then $R \cap R' \in \mathcal{U}$;

(iii) for any $R \in \mathcal{U}$, there exists $R' \in \mathcal{U}$ such that $2R' \subseteq R$, where $2R'$ denotes the set $\{(a,b) \mid \exists c \in X,\ (a,c),\ (c,b) \in R'\}$;

(iv) $\bigcap_{R\in\mathcal{U}} R$ is equal to the diagonal $\{(x,x) \mid \in X\}$.

Any uniformity $\mathcal{U}$ on a set X determines a topology on it as follows: O is open if, for any $x \in O$, there exists $R \in \mathcal{U}$ such that $B(x,R) \subseteq O$; here $B(x,R)$ denotes the set $\{y \mid (x,y) \in R\}$. This topology is completely regular [6].

Applying Lemma 2, Lemma 3, Proposition 8, and Lemma 9, one can prove

Theorem 10. *Let $\mathbb{V}$ be a protomodular variety of universal algebras, and let A be a right-cancellable topological $\mathbb{V}$-algebra that satisfies the separation axiom T_0. Then the topology of A is determined by a uniformity, and hence is completely regular.*

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