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Mathematics

Extremal Problems in Kendall Shape Spaces

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ABSTRACT. We present a number of results concerned with extremal values of several natural functions on Kendall shape spaces of planar polygons. The main attention is given to the oriented area and Coulomb energy. In particular, we investigate in some detail an analog of the classical isoperimetric problem in the setting of Kendall shape spaces. Analogous results are obtained for poristic triangles and Coulomb energy of unit charges placed at the vertices of polygon. *© 2019 Bull. Georg. Natl. Acad. Sci.*

Key words: configuration of points, Kendall shape space, isoperimetric problem, oriented area, Coulomb potential, Atiyah determinant, poristic triangle

Our aim is to find the extremal and critical values of several geometrically and physically meaningful functions on planar shape spaces introduced by D. Kendall [1]. Let us begin with precise description of the setting and related concepts. To define the main object let us denote by $C^*(k, m)$ the set of all configurations of k labelled points x_j in $\mathbb{R}^m$, such that not all of them coincide, and endow it with the natural topology inherited from $\mathbb{R}^m$. Let us denote by $Sim(m)$ the group of similarities of $\mathbb{R}^m$ generated by the parallel shifts, rotations and homotheties. This group has an obvious diagonal action on $C^*(k, m)$ and the *Kendall shape space* $K(k, m)$ is defined as the factor-space of $C^*(k, m)$ over this action of $Sim(m)$. More precisely, $K(k, m)$ is the space of orbits of $Sim(m)$ endowed with the factor-topology. As is well known, $K(k, m)$ is a compact and connected Hausdorff topological space [1]. Moreover, there is a natural distance function on $K(k, m)$ generating the same topology [1]. The topological structure of shape spaces is quite nontrivial and attracted a lot of attention [1]. Nowadays it is well understood in many important cases.

We will basically deal with the *planar shape space* $K(k, 2)$ and think of its elements as shapes of oriented k-gons in the plane. As was shown in [1], the space $K(k, 2)$ is homeomorphic to the complex projective space $\mathbb{CP}^{k-2}$. In fact, the canonical metric on $K(k, 2)$ is equivalent to a properly normalized Fubini-Study metric on $\mathbb{CP}^{k-2}$ [1]. In particular, for $k=3$ the shape space of labelled triangles $K(3, 2)$, is isometric to a sphere of radius $1/2$ in $\mathbb{R}^3$ endowed with the geodesic (great circle) distance [1]. This sphere contains several distinguished subsets corresponding to specific triangular shapes such as regular (poles), isosceles (meridians), aligned (equator), degenerate (3 points on equator) and right (three circles passing through the degenerate shapes and orthogonal to the equator) [1]. In the sequel we discuss several extremal problems on $K(3, 2)$ and describe their relation to the aforementioned subsets of $K(3, 2)$.

In general, any geometric characteristic of planar oriented polygons which is invariant with respect to similarities defines a function F on $K(k, 2)$. So one may wish to find the extremal values of such a function F on $K(k, 2)$ and identify the corresponding points of $K(k, 2)$ called *F-extremal shapes*. The most natural extremal problem on $K(k, 2)$ arises as an analog of the classical isoperimetric problem. For any ordered k-tuple of points in the plane, let us consider the corresponding k-gon P and denote by $A(P)$ and $L(P)$ the oriented area and perimeter of P, respectively. For any not totally degenerate ordered k-tuple of points in the plane, the ratio $A(P)/(L(P))^2$ is well-defined and similarity invariant. So it yields a continuous function $F_1=A/L^2$ on $K(k, 2)$ and one may wonder what are the extremal values of F_1 on $K(k, 2)$. The well known results on the classical isoperimetric problem imply that F_1 attains its extrema at the shape of regular k-gon taken with both possible orientations.

The main novelty of our approach is that, in the spirit of [2], we also consider other critical points of function F_1, which leads to interesting results and serves as a paradigm for further study of extremal problems in the context of shape spaces. To this end we notice that function F_1 is continuously differentiable everywhere except a measure zero subset Z consisting of degenerate shapes with some (but not all) coinciding vertices. Our first main result (Theorem 1) states that all critical points of F_1 outside of Z are the shapes of the so-called *regular stars*. For $k=5$, we show that the two 5-stars are non-degenerate and compute their Morse indices (Proposition 1).

It should be noted that a similar problem in a slightly different setting was studied in a recent joint preprint of the author with G.Panina and D.Siersma [3] (cf. also [4]) and the proof of Theorem 1 essentially relies on the results and arguments from [3] and [4]. Other extremal problems considered in the sequel apparently have not been discussed in the literature. Theorem 1 suggests investigation of similar extremal problems in the context of Kendall shape spaces. Along these lines, for $k\geq 4$, we consider a version of extremal problem studied in [5]. To this end we denote by $D(P)$ the sum of lengths of all pairwise distances between given points (i.e., diagonals and sides of P) and denote by F_2 the function on $K(k,2)$ defined by the expression $D(P)/L(P)$. We find the extrema and extremal shapes for $k=4$ and $k=5$ (Propositions 2 and 3) and present some remarks in the general case.

Another natural function on $K(k, 2)$ arises from electrostatics and was already considered in a similar context in [6]. For a given k-tuple of points P, let us denote by $E(P)$ the Coulomb potential of k identical unit charges placed at the points of P. Then $E(P)=\Sigma 1/d_{ij}$, where d_{ij} is the distance between p_i and p_j. Obviously, the product $E(P)\cdot L(P)$ is similarity invariant and defines a differentiable function $F_3=E(P)\cdot L(P)$ on $K(k, 2)\backslash Z$. Critical points again correspond to regular k-stars (Theorem 3). For $k=5$, the two 5-stars are non-degenerate minima. In fact, the Coulomb potential on $K(k, 2)$ can be included in a one-dimensional family arising from the Riesz s-potentials but the critical points of such functions are more difficult to analyze for $s > 1$.

An interesting example of similarity invariant function on configurations of k points is the so-called *Atiyah determinant* Det_A [7] which we denote by F_4. We do not define it for arbitrary k because in the sequel we consider it only for $k=3$, where it is just equal to the sum of the squared sines of the angles [7]. We also present an analogous result for poristic triangles (Proposition 5).

In the sequel, a *regular k-gon* means an equilateral equiangular *k*-gon. Note that, for *k≥4*, a regular polygon can be convex or self-intersecting. In the latter case, if there are only transversal intersections of the sides, it will be called a regular *k-pointed star* or simply a *regular k-star*. For even *k*, there also exists a degenerate regular *k*-gon, called *k-pile*, with coinciding all odd-numbered vertices and coinciding all even-numbered vertices. In the definition of Kendall shape space and in the sequel it is assumed that any *k*-gon is oriented by the given ordering of its vertices.

Recall that the *winding index* of a regular *k*-gon is an integer number equal to the winding number of its sides about its center. It is easy to see that the absolute value of winding number of *k*-gon belongs to the integer segment [1, [(n-1)/2]], where the inner square brackets denote the integer part of the number. The winding index of a pile is defined to be zero.

To present our results in a rigorous way, we take into account that some of the foregoing functions may not be defined or be non-differentiable at certain points of Kendall shape space. It is easy to verify that these complications may only arise at the shapes of configurations with coinciding points (like the pile). In order to avoid this difficulty we will consider all functions on the open subspace $K_s(k,2)$ which *is* the complement of the subset Z(k,2) consisting of all shapes of n-gons some of vertices of which coincide. For *n=3, Z(k,2)* consists of three points in the equator of the Kendall sphere S^2 corresponding to degenerate triangles with two coinciding vertices. In this case it appears possible to describe the behaviour of function near each of these points. Further remarks on the behaviour of considered functions near points from *Z(k,2)* with *k≥4* will be given at the end of this note. We can now give the precise formulations of the main results.

Theorem 1. *For any k≥3 , the critical points of function* F_1 *on* $K_s(k,2)$ *are given by the shapes of regular k-gons with both possible orientations. The absolute maximum and absolute minimum are attained at the shapes of convex regular k-gons with counter-clockwise and clockwise orientations respectively.*

Proposition 1. *For k=5, the stars are nondegenerate critical points of* F_1*, the Morse index of positively oriented star is 4 and the Morse index of negatively oriented star is 2.*

Analogous results for the oriented area in slightly different contexts have been proven in [2] and [5]. The proofs in our setting are similar to those given in [2] and [5].

Theorem 2. *For any k≥3 , the critical points of function* F_2 *in* $K_s(k,2)$ *are given by the shapes of regular k-stars. Maximum is attained at convex regular k-gon.*

Proposition 2. *For k=4, the square shapes with both orientations are the points of non-degenerate maxima, and the minimum is attained at the 4-pile (4-fold segment).*

Proposition 3. *For k=5, the regular pentagon shapes with both orientations are non-degenerate maxima and the minimum is attained at regular triangle with a three-fold side.*

These results and computations used in their proofs suggest the following conjecture concerned with the function F_2 on the space *K(k,2).*

Conjecture. *For even k, the global minimum of* F_2 *is attained at the k-pile. For odd k, the global minimum of* F_2 *is attained at degenerate regular triangle with one side represented by (k-2)-pile.*

Theorem 3. *Critical points of function* $F_3 = E \cdot L$ *in* $K_s(k,2)$ *are the regular k-stars which are local minima. The global minimum is attained at the regular k-gon. Function* F_3 *has no maxima in* $K_s(k,2)$ *but has poles at each point of Z(k,2).*

This is verified by direct computation of the gradient and hessian at each regular *k*-star.

Proposition 4. (cf. [7]) *The maximum of the Atiyah determinant F_4 on K(3, 2) is attained at the shape of regular triangle and equals $9/8\sqrt{2}$. The minimum is attained at the symmetric aligned configuration and equals $1/\sqrt{2}$.*

As was shown in [7], the Atiyah determinant of a triangle Δ is equal to the sum of squared sines of half-angles of Δ. The result then follows by a simple application of Lagrange multipliers method.

The latter result suggests that, for triangular shapes, one can obtain analogous results by considering various symmetric functions of the angles of a triangle. To illustrate this idea we present one result of such type concerned with poristic triangles provided by Poncelet theorem [8]. Recall that *poristic triangles* are defined as the family of all triangles having the same incenter and circumcenter. Each family of poristic triangles is one-dimensional and contains one tall (the angle at the apex is less than $\pi/2$) and one flat (the angle at the apex is greater or equal to $\pi/2$) isosceles triangle (see, e.g., [8]). The following result is proven by analyzing the extrema of the function F_5 equal to the sum of tangences multiplied by the sum of cotangences of the angles of the given triangle.

Proposition 5. *The maximum of F_1 on poristic n-gons is attained at the tall isosceles triangle. The minimum is attained at the flat equilateral configuration.*

Our results suggest many generalizations and open problems. Omitting the most evident ones, like computing the Morse indices in general setting or considering the spatial shape spaces, we would only like to mention a promising and less obvious direction of research concerned with behaviour of functions F_j near the points of set *Z(k,2)* introduced above. It turns out that one can get further results using a more general concept of critical point suggested in the classical book [9]. For example, it can be proven that the function F_1 does not have critical points in *Z(5,2)* and so it gives an example of exact Morse function on the whole *K(5,2)*. Applying this idea to other functions on Kendall shape spaces seems to be an interesting research perspective.

მათემატიკა

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