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Mathematics

On Algebraic *K*-Functors of Crossed Restricted Enveloping Algebras of Lie *p*-Algebras

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ABSTRACT. A crossed universal Λ-enveloping restricted associative algebra of Lie *p*-algebra *L*, where Λ is a commutative *L*-module algebra, is constructed, and P.B.W. theorem is proved for him. It is proved that such algebras are particular cases of crossed Hopf algebras, its algebraic *K*-functors are Frobenius modules over Grothendieck functor of enveloping algebra of Lie (*p*)-algebra and Sweedler and Hopf cohomologies of *k*[L] with coefficients in commutative algebras are isomorphic. *© 2019 Bull. Georg. Natl. Acad. Sci.*

Key words: crossed enveloping algebra, Lie (p)-algebra, algebraic *K*-functors

In this paper a crossed universal (restricted) Λ -enveloping algebra of Lie (p -)algebra is constructed, where Λ is a commutative algebra over a ground field k , on which Lie (p -)algebra acts by derivations. It is shown that they are particular cases of the crossed Hopf algebras [1], and P.B.W. theorems are proved for them. It is also proved that algebraic *K*-functors [2-4] of the crossed Λ -enveloping algebra of Lie (p -)algebra are Frobenius modules [5] over Grothendieck functor [5] of a suitable restricted universal algebra of Lie p -algebra. These results generalize corresponding results for restricted crossed enveloping algebras of finite dimensional Lie p -algebras from [6]. The following results are relevant to this work: in [7] it was proved that the Grothendieck group of a finite group is a Frobenius functor, in [5] and [8] it was proved that the algebraic K_n -functors of a finite group G , $n \geq 1$ are Frobenius modules over the Grothendieck group of G, which were generalized for twisted group rings in [9, 10].

Let k be a commutative algebra with identity, A an associative k -algebra, $M \in k-mod$. A k -module

$$T(A,M) = A \oplus (A \otimes M \otimes A) \oplus (A \otimes M \otimes A \otimes M \otimes A) \oplus ...$$

is an associative algebra with multiplication

$$(a_1 \otimes m_1 \otimes ... \otimes n_1 \otimes b_1)(a_2 \otimes m_2 \otimes ... \otimes n_2 \otimes b_2) =$$
$$= a_1 \otimes m_1 \otimes ... \otimes n_1 \otimes b_1 a_2 \otimes m_2 \otimes ... \otimes n_2 \otimes b_2 .$$

Let us construct mappings $i_1 : A \to T(A,M), a \to a,\ i_2 : M \to T(A,M), m \to 1 \otimes m \otimes 1.$

This triplet $(T(A,M),i_1,i_2)$ has the property of universality: if B is an associative algebra, $f_1 : A \to B$ is a morphism of associative algebras, $f_2 : M \to B$ is a morphism of *k*-modules, then there exist an unique morphism of algebras $f : T(A,M) \to B$ such that $Fi_1 = f_1, Fi_2 = f_2$.

Further we assume that k is a field and *char k=p* in the case of a Lie *p*-algebra. Let L be a Lie (p)-algebra with a *p*-mapping $x \to x^{[p]}$ [11]. Suppose L acts by derivations on a commutative k-algebra Λ. Λ_p denotes the Lie *p*-algebra with Λ as an underlying space, with a zero multiplication and a p-mapping $a^{[p]} = a^p$.

Let $[(\alpha,\beta)] \in \bar{H}^2(L,\Lambda_p)$, $\alpha : L \otimes L \to \Lambda$, $\beta : L \to \Lambda$, be a 2-cocycle of Pareigis cohomology of Lie p-algebra [11] of L with coefficients in Λ_p. Then

$$\alpha(x,x)=0,$$

$$\alpha(x_0 \otimes [x_1,x_2]) + \alpha(x_1 \otimes [x_2,x_0]) + \alpha(x_2 \otimes [x_0,x_1]) + x_0\alpha(x_1 \otimes x_2) + x_1\alpha(x_2 \otimes x_0) + x_2\alpha(x_0 \otimes x_1) = 0,$$

$$\beta(\lambda x) = \lambda^p \beta(x), \quad \alpha(x^{[p]} \otimes x') - x'\beta(x) = \sum_{i=o}^{p-1} x^{p-1-i}\alpha(x \otimes (adx)^i(x')) = \Gamma(x,x') \tag{1}$$

$$\beta(x_0 + x_1) - \beta(x_0) - \beta(x_1) = \sum_t \sum_{i=1}^{p-1} \frac{1}{|t^{-1}(0)|} x_{t(1)} \dots x_{t(i-1)} \alpha(x_{t(i)} \otimes [x_{t(i+1)}[\dots[x_{t(p-1)}, x_1]\dots]].$$

First two conditions in (1) define a 2-cocycle $[\alpha] \in H^2(L,\Lambda^+)$ of second cohomology group of the Lie algebra L with coefficients in the ground space Λ^+ of Λ.

Definition 1. a) A crossed Λ-enveloping algebra of a Lie algebra L with respect to the 2-cocycle $[\alpha] \in H^2(L,\Lambda^+)\backslash$ is a triple $(\Lambda(L,\alpha),i_1,i_2)$ where

$$\Lambda(L,\alpha) = T(\Lambda,L) / \{1 \otimes x \otimes \lambda - \lambda \otimes x \otimes 1 - x \cdot \lambda, 1 \otimes x_0 \otimes 1 \otimes x_1 \otimes 1 - 1 \otimes x_1 \otimes 1 \otimes x_0 \otimes 1 -$$
$$-1 \otimes [x_0,x_1] \otimes 1 - \alpha(x_0 \otimes x_1)\}$$

and i_1,i_2 are induced by the corresponding embedding of Λ and L in $T(\Lambda,L)$.

b) A crossed Λ-enveloping restricted algebra of a Lie *p*-algebra L with respect to the 2-cocycle $[\alpha] \in \bar{H}^2(L,\Lambda_p)$ is a triple $(\Lambda[L,\alpha,\beta],i_1,i_2)$, where

$$\Lambda[L,\alpha,\beta] = \Lambda(L,\alpha) / \{x^p - x^{[p]} - \beta(x)\}$$

and i_1,i_2 are induced by the corresponding embeddings of Λ and L in $T(\Lambda,L)$.

Instead of $i_1(\lambda)$ we will write λ. We denote $i_2(x)$ by $\bar{x}$ if $[\alpha] \neq 0\backslash$ or $[(\alpha,\beta)] \neq 0$ and by x if $[\alpha]=0$ or $[(\alpha,\beta)]=0$.

An algebra $\Lambda(L,\alpha)$ was defined in [12] for graded Lie algebras in the case $\Lambda = k$. An algebra $\Lambda[L,\alpha,\beta]$ was defined in [13] in the case $\Lambda = k, \alpha = 0$.

The following proposition is easy:

Proposition 2. a) Let L be a Lie algebra. If (B,f_1,f_2) is a triplet, B is an associative algebra, $f_1 : \Lambda \to B, f_2 : L \to B$ are morphisms of algebras and k-modules, and

$$f_2(x)f_1(\lambda) = f_1(b)f_2(x) + f_1(x \cdot \lambda), \quad f_2([x_0,x_1]) = [f_2(x_0),f_2(x_1)] + f_1\alpha(x_0 \otimes x_1), \tag{2}$$

then there exists a unique morphism of algebras $F:\Lambda(L,\alpha)\to B$ such that $Fi_1=f_1, Fi_2=f_2$.

b) Let L be a Lie p-algebra and (B,f_1,f_2) be the triplet such that $f_2(x^{[p]})=f_2(x)^p+f_1\beta(x)$. Then there exists a unique morphism of algebras $F:\Lambda[L,\alpha,\beta]\to B$ such that $Fi_1=f_1, Fi_2=f_2$.

Let $k(L)$ denotes an enveloping algebra of a Lie algebra L and $k[L]$ denotes a restricted enveloping algebra of a Lie p-algebra L. Then $k(L)$ and $k[L]$ are Hopf algebras [14] with comultiplication Δ an antipode S such that $S^2=id$ and $Hom(k(L),\Lambda(L,\alpha))$ is an associative algebra by multiplication $(f*g)(c)=\mu(f\otimes g)\Delta(c)$. Similarly for Lie p–algebras.

Proposition 3. $\Lambda(L,\alpha)$ ($\Lambda[L,\alpha,\beta]$) is a comodule algebra [14] over $k(L)$ ($k[L]$).

Proof. Let us define $\chi:L\to\Lambda(L,\alpha)$ as $\chi(x)=\overline{x}\otimes 1+1\otimes x$. It is easy to verify that conditions (2) are true, therefore by Proposition 2 the mapping χ can be extended on $\Lambda(L,\alpha)$. The case of a Lie p-algebra is proved similarly.

Proposition 4. a) There exist an invertible morphism of $k(L)$-modules $\gamma\in Hom(k(L),\Lambda(L,\alpha))$.

b) There exist an invertible morphism of $k[L]$-modules $\gamma\in Hom(k[L],\Lambda[L,\alpha,\beta])$.

Proof. a) Let $\{x_i\}_{i\in I}$ be a basis of L where I is a completely ordered set. By P.B.W. theorem a set $\{x_{i_1}x_{i_2}...x_{i_n}\}$, $i_1\le i_2\le...\le i_n$ is a basis of $k(L)$. Let us define

$$\gamma:k(L)\to\Lambda(L,\alpha),\quad x_{i_1}...x_{i_n}\to\overline{x}_{i_1}...\overline{x}_{i_n}.$$

It is clear that γ is a morphism of $k(L)$-modules. Let us prove that $\gamma*\gamma S=\eta\varepsilon$:

$$(\gamma*\gamma)(x_{i_1}x_{i_2}...x_{i_n})=\mu(\gamma\otimes\gamma S)\sum_{r_i+s_i=1}x_{i_1}^{r_1}...x_{i_n}^{r_n}\otimes x_{i_1}^{s_1}...x_{i_n}^{s_n}=\sum_{r_i+s_i=1}\overline{x}_{i_1}^{r_1}...\overline{x}_{i_n}^{r_n}(-1)^{s_1+...+s_n}\overline{x}_{i_n}^{s_n}...\overline{x}_{i_1}^{s_1}=$$

$$=\sum_{r_i+s_i=1}\overline{x}_{i_1}^{r_1}...\overline{x}_{i_{n-1}}^{r_{n-1}}\overline{x}_{i_n}(-1)^{s_1+...+s_{n-1}}\overline{x}_{i_{n-1}}^{s_{n-1}}...\overline{x}_{i_1}^{s_1}+\sum_{r_i+s_i=1}\overline{x}_{i_1}^{r_1}...\overline{x}_{i_{n-1}}^{r_{n-1}}(-1)^{s_1+...+s_{n-1}+1}\overline{x}_{i_n}\overline{x}_{i_{n-1}}^{s_{n-1}}...\overline{x}_{i_1}^{s_1}=0.$$

Also $(\gamma*\gamma S)(1)=1$, that is $\gamma S=\gamma^{-1}$. The case b) is proved similarly.

Theorem 1. Let $\{x_i\}_{i\in I}$ be a basis of L where I is a completely ordered set. Then

a) $\Lambda(L,\alpha)$ is a free left (also right) module over Λ with the basis

$\{\overline{x}_{i_1}^{r_1}...\overline{x}_{i_{n-}}^{r_n}\}$, $i_1<i_2<...<i_n$.

b) $\Lambda[L,\alpha,\beta]$ is a free left (also right) module over Λ with the basis

$\{\overline{x}_{i_1}^{r_1}...\overline{x}_{i_{n-}}^{r_n}\}$, $i_1<i_2<...<i_n, 0\le r_i<p$.

Proof. From Definition 1 it follows that the above mentioned elements are generators of $\Lambda(L,\alpha)$. Let us prove that they are linearly independent. Let us define

$$P(\chi)\otimes k(L)\to\Lambda(L,\alpha),\quad au\to a\gamma(u),\tag{3}$$

where $P(\chi)=\{u\in\Lambda(L,\alpha)\mid\chi(u)=u\otimes 1\}$ contains Λ and is the subring of $\Lambda(L,\alpha)$ and γ is the morphism from the proof of Proposition 3. As in Lemma 8.3, [14] (see also Lemma 1) it is easy to show that the mapping $u\to\sum_u u_{(0)}\gamma^{-1}(u_{(1)})\otimes u_{(2)}$ is the inverse to (3). Therefore (3) is an isomorphism of left $P(\chi)$-modules. Since $1\otimes x_{i_1}^{r_1}...x_{i_n}^{r_n}\to\overline{x}_{i_1}^{r_1}...\overline{x}_{i_{n-}}^{r_n}$, $i_1<i_2<...<i_n$, and by P.B.W. theorem $x_{i_1}^{r_1}...x_{i_n}^{r_n}$ are linearly independent, $\overline{x}_{i_1}^{r_1}...\overline{x}_{i_n}^{r_n}$ are linearly independent over $P(\chi)$ and therefore over Λ. The case b) is proved similarly.

Theorem 1(a) was proved in [12] for graded Lie algebras when $\Lambda = k$. Theorem 1(b) was proved in [13] when $\Lambda = k, \alpha = 0$.

Proposition 5. Let $u \in \Lambda(L,\alpha)$ ($u \in \Lambda[L,\alpha,\beta]$) and $\chi(u) = \sum_{(u)} u_{(0)} \otimes u_{(1)}$. Then $u\lambda = \sum_{(u)} (u_{(1)}\lambda)u_{(0)}$.

Proof. If $u = \overline{x}$, the statement follows from Definition 1. The general case is proved by induction on the length of u.

Proposition 6.

$$a)\ \Lambda = \{u \in \Lambda(L,\alpha) \mid \chi(u) = u \otimes 1\}, \quad b)\ \Lambda = \{u \in \Lambda[L,\alpha,\beta] \mid \chi(u) = u \otimes 1\}. \tag{4}$$

Proof. Let $u = \sum_I \lambda_I \overline{x}^I$ where $I = \{i_1, i_2, ..., i_n\}, i_1 \le i_2 \le ... \le i_n$, $\overline{x}^I = \overline{x_1}^{i_1} ... \overline{x_{i_n}}^{i_n}$. If u satisfies (4) then

$$\sum_I \lambda_I \overline{x}^I \otimes 1 = \sum_I \sum_J \lambda_I \overline{x}^J \otimes x^{I \setminus J}. \tag{5}$$

By Theorem 1 elements $\overline{x}^J \otimes x^{I \setminus J}$ are linearly independent over Λ. If $I \ne \varnothing$ then from (5) it follows that $\lambda_I = 0$, therefore $u \in \Lambda$.

Sweedler cohomology [14] $H^n(H,\Lambda)$ of a Hopf algebra H and a H–module commutative algebra Λ with action $\psi : H \otimes \Lambda \to \Lambda$ are homology groups of the complex $d^{q-1} : Reg(\otimes^q H, \Lambda) \to Reg(\otimes^{q+1} H, \Lambda)$, $q = 1,2,...$, where $Reg(\otimes^q H, \Lambda)$ are invertible elements of $Hom(\otimes^q H, \Lambda)$ with respect to the product * and

$$d^{q-1}(f) = [\psi(I \otimes f)] * [f^{-1}(\mu \otimes I \otimes ... \otimes I)] * [f(I \otimes \mu \otimes ... \otimes I)] * ...$$

$$* [f^{\pm}(I \otimes I \otimes ... \otimes \mu)] * [f^{\mp} \otimes \varepsilon].$$

Let $\sigma \in Reg_+^2(H,\Lambda) = \{ f \in Reg^2(H,\Lambda) | (\exists i) h_i \in k \Rightarrow f(h_1,h_2) = \varepsilon(h_1)\varepsilon(h_2) \}$. Then a crossed product $\Lambda \#_\sigma H$ of a Hopf algebra H and a H–module commutative algebra Λ is [14] an associative algebra with the underlying space $\Lambda \otimes H$ and with a multiplication

$$(\lambda \#_\sigma g)(\mu \#_\sigma h) = \lambda \sum_{g,h,} (g_{(1)}\mu)\sigma(g_{(2)} \otimes h_{(1)}) \otimes g_{(3)} h_{(2)}.$$

Theorem 2. For a Lie p-algebra L there exists a natural isomorphism of Sweedler cohomologies [14] of Hopf algebra $k[L]$ with coefficients in a $k[L]$-module [14] commutative algebra Λ and Hochschild cohomologies of L with coefficients in Λ^+

$$H^n(k[L], \Lambda) \simeq H^n(L, \Lambda^+), \quad n \ge 2.$$

Proof. A normal complex corresponding to $H^n(L, \Lambda^+)$ consists of modules

$$C_n = \{ f \in Hom(\otimes^n k[L], \Lambda) \mid f(\lambda_1 \otimes ... \otimes \lambda_n) = 0 \quad \text{if} \quad \lambda_i \in k \quad \text{for some} \quad i \},$$

where $n > 0, C_0 = \Lambda^+$ with corresponding derivations. If we define a structure of $k[L]$-module on $Hom(k[L], \Lambda)$ as $(l \rightharpoonup f)(u) = f(ul)$, $l \in L$ then $Hom(k[L], \Lambda)$ becomes a $k[L]$-module algebra, i.e.

$$u \rightharpoonup (f * g) = \sum_u (u_{(1)} \rightharpoonup f) * (u_{(2)} \rightharpoonup g).$$

It is clear that the projection $k(L) \to k[L]$ induces the $k(L)$-module structure on $Hom(k[L], \Lambda)$. For $f \in C_1$ let us define a sequence $f^{(0)}, f^{(1)}, f^{(2)}, \ldots \in C_1$ as $f^{(0)} = \eta\varepsilon$, $f^{(1)} = f$ and suppose that there are already defined $f^{(i)}, i < n$. Let $Y = \{ky\}$ be a one-dimensional vector space. Then $Hom(k[L], \Lambda) \oplus Y$ is a $k(L)$-module by action $l \rightharpoonup (\varphi, \lambda y) = l \rightharpoonup \varphi + f^{(n-1)} * (l \rightharpoonup f), l \in L.$ Let us prove $l^p(\varphi, \lambda y) = l^{[p]}(\varphi, \lambda y).$ It is sufficient to consider the case $\varphi = 0, \lambda = 1$. We have

$$l^p(0, y) = \sum_{r_0 + 2r_1 + \ldots + qr_{q-1} = p}^{r_0 + r_1 + \ldots + r_{q-1} = i} a_r f^{(n-i)} * (l \rightharpoonup f)^{r_0} * (l^2 \rightharpoonup f)^{r_1} * \ldots * (l^q \rightharpoonup f)^{r_{q-1}},$$

where degree is calculated with respect to $*$. Using induction we can verify that

$$a_{r_1, \ldots, r_{q-1}} = \left[\prod_{j=1}^{q-1} \prod_{i=1}^{r_j} \binom{ij + i - 1}{p} \right] \binom{p}{2r_1, 3r_2, \ldots, qr_{q-1}}. \tag{6}$$

Therefore $l^p(0, y) = f^{(n-i)} * (l - f)^p + f^{(n-1)} * (l^p - f)$, because all other coefficients are divisible by p.

Since $char(k) = p$, then $\varphi^p = 0$ for any $\varphi \in Hom(k[L], \Lambda)$. Therefore

$$l^p(0, y) = f^{(n-i)} * (l - f)^p = f^{(n-1)} * (l^{[p]} - f) = l^{[p]}(0, y).$$

Consequently this action induces a structure of $k[L]$-module on $Hom(k[L], \Lambda) \oplus Y$. Since $k[L] = Ker\varepsilon \oplus k$ we may define $f^{(n)}(1) = 0, f^{(n)}(u) = (u \rightharpoonup y)(1)$ if $u \in Ker(\varepsilon)$. From the definition it follows that

$$l \rightharpoonup f^{(n)} = f^{(n-1)} * (l \rightharpoonup f). \tag{7}$$

The constructed sequence is uniquely defined by (7) and the conditions $f^{(0)} = \eta\varepsilon, f^{(1)} = f, f^{(n)} = 0,$ if $n > 1$. The next proof is similar to the proof of the analogous result for Lie algebras in [14] (Theorem 4.3). Namely, for any $u \in k[L]$ there exist i such that $n > i \Rightarrow f^{(n)}(u) = 0$. Therefore one can prove that morphisms

$$exp : C_1 \to Reg^1_+(k[L], \Lambda), \quad exp(f) = \sum_{i=0}^{\infty} f^{(i)},$$

$$log : Reg^1_+(k[L], \Lambda) \to C_1, \quad log(f + \eta\varepsilon) = \sum_{i=0}^{\infty} i!(-1)^i f^{(i+1)}$$

are the group isomorphisms. Using these mappings, we construct the isomorphism $C_n \to Reg^n_+(k[L], \Lambda)$ which commutes with derivations.

Proposition 7. A subspace $E = \Lambda \#_\sigma 1 + 1 \#_\sigma L \subseteq \Lambda \#_\sigma k[L]$ is a Lie p-algebra with respect to the multiplication [,] and the p-mapping $u \to u^p$. This statement is valid also for Lie algebras.

Proof. Let $x, y \in L$. We have:

$$[\lambda+1\#_\sigma x,\mu+1\#_\sigma y]=\lambda\mu+\lambda\#_\sigma y+x\cdot\mu+\mu\#_\sigma y+\sigma(x\otimes y)+$$
$$+1\#_\sigma xy-\mu\lambda-\mu\#_\sigma x-y\lambda-\lambda\#_\sigma y-\sigma(y\otimes x)-1\#_\sigma yx= \tag{8}$$
$$=1\#[xy]+\sigma(x\otimes y)-\sigma(y\otimes x)+x\mu-y\lambda\in E\,.$$

It is easy to prove by induction that $(\lambda+1\#_\sigma x)^p=\sum_{i=0}^{p}\binom{p}{i}\lambda^i\#_\sigma x^{p-i}(mod\Lambda)$, from which it follows that $(\lambda+1\#_\sigma x)^p\in E$. Since $\Lambda\#_\sigma k[L]$ is a Lie p-algebra with respect to the operations $[-,-],(-)^p$, then E is a Lie p-algebra. In addition, let us remark that the coefficient of $\sigma(x\otimes x)^{r_1}\sigma(x^2\otimes x)^{r_1}...\sigma(x^{q-1}\otimes x)^{r_{q-1}}\#_\sigma x^{r_0}$ in $(1\#_\sigma x)^p$ is expressed by (6). Consequently $(1\#_\sigma x)^p=1\#_\sigma x^{[p]}+\sigma(x^{p-1}\otimes x)$.

Corollary. Let us define $\alpha(x\otimes y)=\sigma(x\otimes y)-\sigma(y\otimes x),\ \beta(x)=\sigma(x^{p-1}\otimes x), x,y\in L$.
Then the pair (α,β) is a 2-cocycle of Pareigis cohomology of Lie p-algebras.
Proof. We must prove that the pair (α,β) satisfies conditions (1). This is because E is a Lie p-algebra. For example, let us prove the fourth condition of (1). Indeed, from (8) it follows that $ad(1\#_\sigma x)(1\#_\sigma y)=1\#_\sigma ad(x)(y)+\alpha(x,y)$. Thus $ad(1\#_\sigma x)^p)(1\#_\sigma y))=1\#_\sigma ad(x)^p(y)+\Gamma(x,y)$. On the other hand $ad(1\#_\sigma x)^p)(1\#_\sigma y))=ad(1\#_\sigma x^{[p]}+\beta(x))(1\#_\sigma y)=1\#_\sigma[x[p],y]+\alpha(x[p]\otimes y)-y\beta(x)$, because $(ad1\#_\sigma x)^p=ad((1\#_\sigma x)^p)$.

Theorem 3. *i)* $\Lambda(L,\alpha)$ and $\Lambda[L,\alpha,\beta]$ are splitting extensions of $k(L)$ and k[L] respectively.
ii) $\Lambda(L,\alpha)\simeq\Lambda(L,\alpha')$ as the extensions if and only if $\alpha\sim\alpha'$,
$\Lambda[L,\alpha,\beta]\simeq\Lambda[L,\alpha',\beta']$ as the extensions if and only if $(\alpha,\beta)\sim(\alpha',\beta')$.
iii) $\Lambda\#_\sigma k(L)\simeq\Lambda(L,\alpha)$ and $\Lambda\#_\sigma k[L]\simeq\Lambda[L,\alpha,\beta]$ for some α and (α,β) respectively.

Proof. Let us consider only the case of Lie p-algebras.

i) This follows from the propositions 3 and 6.
ii) Let us recall that $(\alpha,\beta)\sim(\alpha',\beta')$ means

$$(\alpha'-\alpha)(x\otimes y)=x\psi(y)-y\psi(x)-\psi([xy]),\ (\beta-\beta')(x)=\psi(x)^p+x^{(p-1)}\psi(x)-\psi(x[p]),$$

where $\psi\in Hom(L,\Lambda)$. It is clear that if $(\alpha,\beta)\sim(\alpha',\beta')$ then $L\to\Lambda[L,\alpha,\beta]$, $x\to\overline{x}+\beta(x)$ induces the needed isomorphism. On the other hand, let us consider

$$f:\Lambda[L,\alpha,\beta]\to\Lambda[L,\alpha',\beta'],\quad f|_\Lambda=id\ ,$$

where f is the comodule morphism. We have for $x\in L$

$$(f\otimes id)\chi(\overline{x})-\chi(\tilde{x})=f(\overline{x})\otimes 1+1\otimes x-\tilde{x}\otimes 1-1\otimes x=(f(\overline{x})-\tilde{x})\otimes 1,$$

where $\tilde{x}$ is the image of x in $\Lambda[L,\alpha',\beta']$. From Proposition 6 follows that $f(\overline{x})-\tilde{x}=\psi(x)\in L$. If we calculate $f(u)$ and $f(v)$ for $u=[\overline{x},\overline{y}]-\overline{[xy]}-\alpha[xy]=0$, $v=\overline{x}^p-\overline{x^{[p]}}-\beta(x)=0$, we get the desired identities.

iii) Let us define $\alpha(x,y)=\sigma(x\otimes y)-\sigma(y\otimes x),\quad \beta(x)=\sigma(x^{p-1}\otimes x)$. By Corollary of Proposition 7 the morphisms α and β define an algebra $\Lambda[L,\alpha,\beta]$. As was already shown in the proof of Proposition 7 an embedding $L\to\Lambda\otimes_\sigma k[L]$ satisfies Conditions 5 and 6. Therefore we may extend this embedding to $\Lambda[L,\alpha,\beta]$, which is the isomorphism by Lemma 8.5 from [14].

Let $\bar{H}^n(L,\Lambda_p^+)$ be Pareigis cohomology groups [11] of L with coefficients in Λ_p^+. It is known [11] that $\bar{H}^n(L,\Lambda_p^+) \simeq H^n(L,\Lambda^+), \quad n \geq 3.$

Corollary. Let L acts as the Lie algebra on a commutative algebra Λ. Then

$$H^2(k[L],\Lambda) \simeq \bar{H}^2(L,\Lambda_p^+) \simeq \bar{H}^2(L,\Lambda^+),$$

and these isomorphisms commute with appropriate extensions.

Proof. It follows from Theorem 3 and from the identities

$$(log\sigma)(x \otimes y) = \sigma(x \otimes y), \quad (log\sigma)(x^{p-1} \otimes x) = \sigma(x^{p-1} \otimes x).$$

Let us consider now multiplications of algebraic K-functors of $\Lambda[L,\alpha,\beta]$.

The rings $k(L)$ and $k[L]$ are a pointed Hopf algebras [15]. $\Lambda[L,\alpha,\beta]$ is the crossed product of Λ and Hopf algebra $\Lambda_0[\pi]$, where $\Lambda_0 = \Lambda^{\pi}$. Let us consider the category of groups $G(\pi)$ and the category $G(L)$ of Lie p-subalgebras of L.

Proposition 4. If a Lie p-algebra L acts on a commutative algebra Λ by derivations and $dim(L) < \infty$, then $G_0(k[-])$ is a Frobenius functor and $K_n\Lambda[-,\alpha,\beta])$ are Frobenius modules over $G_0(k[-])$ on the category $G(L)$.

Proof. Proposition is a particular case of Theorem 1.1 from [1] and Theorem 3. The special case of the Theorem 3 when $\Lambda = k$ was proved in [6].

მათემატიკა

ლის *p*-ალგებრის ჯვარედინი შემომფარგვლელი შეზღუდული ალგებრის ალგებრული *K*-ფუნქტორების შესახებ

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L ლის *p*-ალგებრისთვის აგებულია ჯვარედინი უნივერსალური Λ-შემომფარგვლელი შეზღუდული ასოციაციური ალგებრა (Λ არის *L*-მოდულური კომუტაციური ალგებრა), მისთვის დამტკიცებულია პ.ბ.ვ. თეორემა და ნაჩვენებია, რომ ის ჯვარედინი ჰოპფის ალგებრის კერძო შემთხვევაა, ხოლო მისი ალგებრული *K*-ფუნქტორები არის ფრობენიუსის მოდულები *L*-ის მომვლები შეზღუდული ალგებრის გროტენდიკის ფუნქტორის მიმართ. დამტკიცებულია, რომ *k*[L]-ის სვიდლერის და ჰოპფის კოჰომოლოგიები კოეფიციენტებით კომუტაციურ ალგებრებში იზომორფულია.

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