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Mathematics

Isoperimetric Duality in Polygon Spaces

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(Presented by Academy Member Revaz Gamkrelidze)

We discuss interpretations of regular star polygons as solutions to certain dual extremal problem in spaces of polygons. In particular, we describe several situations where regular star polygons appear as non-degenerate critical points and indicate some relations between the Morse indices of the problem and its dual. The main attention is given to the classical isoperimetric problem and its dual in the moduli space of n-gons. Analogous results are given for the class of cyclic polygons with a given circumscribing circle. Some corollaries and possible generalizations are also presented. *© 2020 Bull. Georg. Natl. Acad. Sci.*

Polygon space, moduli space, isoperimetric problem, dual extremal problem, convex regular polygon, regular star polygon, winding number, signed area, perimeter, non-degenerate critical point, Morse index, area foliation

The main aim of this note is to present several observations concerned with a version of isoperimetric problem in the spaces of planar polygons and its dual which can be derived from the recent results presented in [1, 2] using a general approach in the spirit of Morse theory. One of the central ideas of our approach is comparison of Morse indices of common critical points of an extremal problem and its dual, which arises as a generalization of the fundamental duality between the area and perimeter in the classical isoperimetric problem [3].

Recall that one of the classical results on isoperimetric problem states that the regular n-gon has the maximal area among all n-gons with fixed perimeter [3]. In other words, it can be interpreted as a critical point of area considered as a differentiable function on the space of polygons with fixed perimeter endowed with a natural smooth structure. As was recently proven in [1, 2] similar interpretation is possible for regular star n-gons. More precisely, it was shown that they are non-degenerate critical points of area on the space of isoperimetric n-gons and their Morse indices have been computed [2]. An analogous result was obtained in [4] but without discussing non-degeneracy and Morse indices (cf. also [5]). Furthermore, results obtained in [6,7] show that regular star n-gons can also be interpreted as critical points of oriented area on the moduli space of n-gons with prescribed lengths of the sides. These interpretations and related problems have been partially summarized in [1].

The main aim of this paper is to elaborate upon the aforementioned results and present further developments in the same spirit. In particular, we describe a new interpretation of regular star polygons arising from the dual isoperimetric problem. To this end we observe that regular star *n*-gons are critical points of perimeter considered as a function on the set of *n*-gons with fixed oriented area (Theorem 1). Next, using a general result on Morse indices of dual extemal problems (Theorem 2) we compute the homology groups of area levels (Theorem 3), which can be considered as a concrete illustration of possible applications of isoperimetric duality in our setting. We also present several corollaries and analogs of the main result. In particular, we present an analogous result for the set of cyclic *n*-gons with a given circumscribing circle follows from the recent resuls of D. Siersma on the isoperimetric problem for cyclic polygons (Theorem 4). In conclusion it is outlined how the approach and results of this paper can be used to investigate level sets of other natural functions on polygon spaces.

For our purposes it is natural and convenient to work in the context of so-called *polygon spaces* defined as follows. A non-degenerate polygon in $\mathbb{R}^2$ is an element $P=(M_1,...,M_n)$ of $\mathbb{R}^{2n}$ such that: (i) for $i \neq j$ the point M_i is distinct from M_j; (ii) for any point $k \in \{1,...,n\}$, the oriented angle $\hat{M}_k = (\overrightarrow{M_k M_{k+1}}, \overrightarrow{M_k M_{k-1}})$ has its measure in $]0,2\pi[\setminus\{\pi\}$.

The set of all n-gons in the plane $\mathbb{R}^2$ will be denoted $\tilde{\Sigma}_n$. The orbits of the group of the affine isometries of $\mathbb{R}^2$ acting on $\tilde{\Sigma}_n$ are called *geometric polygons* of $\mathbb{R}^2$ and the set of orbits Σ_n is called the *polygon space* (sometimes also called the *moduli space of n-gons*). The polygon space comes equipped with the natural metric and differential structure inherited from $\tilde{\Sigma}_n$. The same object arises in other settings as well and we will also use other interpetations in the sequel. In particular, it is easy to verify that Σ_n is homeomorphic to a cone on the set of all planar *n*-gons with fixed perimeter which is homemorphic to $\mathbb{C}P^{n-2}$ [2]. It can also be identified with the Cartesian product of Kendall shape space [8] with the set of positive real numbers (cf. [9]). Obviously, the functions *p*, perimeter, and *a*, oriented area, defined on nondegenerate polygons in $\mathbb{R}^2$ yield two functions on the polygon space Σ_n.

An *n*-gon $P=(M_1,...,M_n)$ in $\mathbb{R}^2$ is convex if for any $k \in \{1,...,n\}$, the oriented angle $\hat{M}_k = (\overrightarrow{M_k M_{k+1}}, \overrightarrow{M_k M_{k-1}})$ has its measure in $]0,\pi[$. It is called *star-shaped polygon* with respect to the vertex M if for any vertex $N \in \{M_1,...,M_n\}$ the open segment $]M,N[$ is contained in the interior of *P*. In other words, such a polygon can be represented as the union of rays emanating from vertex *M*. For the sake of brevity, we call such polygons *conical* polygons.

For given positive real numbers *r*, *s*, by $p^{-1}(r)=F_r$ $p^{-1}(r) = F_r$ and $a^{-1}(s)=G_s$ we denote the level manifolds of the perimeter and area functions on the polygon space Σ_n. For convenience, we will call them simply *perimeter levels* and *area levels* in Σ_n. These sets can also be referred to as *r-isoperimetric* polygons and *s-isochoric* polygons, respectively. The decompositions of the polygon space Σ_n into disjoint unions of perimeter levels and area levels are called the perimeter and area foliations, respectively. Notice that if we consider the space of *oriented* polygons as a double-cover over Σ_n, then perimeter becomes an even function with non-negative values, while oriented area becomes and *odd* function, which takes negative values as well. Notice that, for any convex polygon *P,* the values of usual (geometric) area $s(P)$ and oriented area coincide, i.e. $s(P)=|a(P)|$, so on the subspace of convex polygons the isoperimetric problem for oriented area is the same as for the classical (non-negative) area function.

A geometric polygon in $\mathbb{R}^2$ is said to be *equilateral* if it admits a representative which is equilateral. In the sequel, a *regular* n*-gon* means an equilateral equiangular n-gon. Note that, for

$n \geq 5$, a regular polygon can be convex or self-intersecting. In the latter case, if there are only transversal intersections of the sides, it will be called a regular *star* n *-gon* or simply a *regular* n *-star*. For even n, there also exists a degenerate regular n-gon, called n *-pile*, with coinciding all odd-numbered vertices and coinciding all even-numbered vertices. Here and below we always assume that any *n*-gon is oriented by the given ordering of its vertices so its oriented area is well defined by the usual *shoelace formula* [3].

Recall that the *winding index* of an oriented n-gon is an integer number equal to the winding rnumber of n-gon belongs to the integer segment $[1,[(n-1)/2]]$, where the inner square brackets denote the integer part of the number. The winding index of a pile is defined to be zero. We can now give precise formulations of the main results. The following result is only formulated for $n \geq 5$ since its analogs for $n=3$ and $n=4$ are quite simple since there are no star polygons in those two cases.

Theorem 1. *For any* $n \geq 5$*, the critical points of perimeter on the set of a-isochoric n-gons are given by regular star* n *-gons of area a taken with both possible orientations. Each such polygon is a non-degenerate critical point. The absolute minimum is attained at the convex regular* n *-gons with both orientations and the absolute maximum is attained at the regular star* n *-gon with the maximal winding index equal to* $[(n-1)/2]$ *and oriented area* A.

This theorem is proved by dualizing the results of [2, 5] using Lagrange multiplier method and results of [8]. The proof essentially relies on a general result given below and results of [2]. Having this result it is natural to wonder what are the Morse indices of regular star *n*-gons as critical points of perimeter on the space of isochoric *n*-gons.

The Morse indices considered can be computed using the following result concerned with a quite general situation. Suppose two smooth real-valued functions f, g are defined on a smooth compact manifold M of dimension $n>1$ and their level surfaces define two foliations F_f and F_g in M. Suppose also that all tangencies between level surfaces of f and g are of finite multiplicity. From Lagrange method it follows that each tangency point $p \in \{f=a\} \cap \{g=b\}$ of level surfaces $\{f=a\}$ and $\{g=b\}$ is a critical point for the restriction $f|\{g=b\}$ and for $g|\{f=a\}$. Denote by $\lambda(p)$ the Lagrange multiplier for $f|\{g=b\}$ at point p. Then the Lagrange multiplier for $g|\{f=a\}$ at point p is $\lambda(p)^{-1}$. The relation between the properties of critical point p in both settings are indicated in the following theorem.

Theorem 2. *The multiplicity of p in both settings is the same. In particular, p is a non-degenerate critical point of* $f|\{g=b\}$ *if and only if p is a non-degenerate critical point of* $g|\{f=a\}$. *If* $\mu(p)$ *is the Morse index of* $f|\{g=b\}$ *at p and* $\lambda(p)>0$ *then the Morse index of* $g|\{f=a\}$ *at p equals n -* $\mu(p)$ and if $\lambda(p)<0$ *then the Morse index of* $g|\{f=a\}$ *at p equals* $\mu(p)$.

This theorem is proved using the natural duality between the target function and constraint function in the setting of Lagrange mutipliers method. The result on non-degenerate critical point is obtained using the extended Hessian and its properties established in [10]. This theorem or some of its statements may be already known but we cannot find it in the literature and present it for convenience of the reader and reference in the sequel.

Corollary 1. In the conditions of theorem 2, a non-degenerate maximum (resp. minimum) point for $f|\{g=b\}$ is a nondegenerate minimum (resp. maximum) point for $g|\{f=a\}$.

Notice that this implies the well known results on isoperimetric problem and its dual in polygon spaces [3, 4].

Moreover, our results can be used to obtain topological information on the leaves of area fibration. In particular, combining Theorem 1 with Theorem 2 and results of [2] we obtain the following general result.

Theorem 3. *For any $n \geq 5$ and $a \neq 0$, the homology groups of the leaf G_a with integer coefficients are as follows: all odd-dimensional homology groups are trivial and all even-dimensional homology groups are isomorphic to $\mathbb{Z}$.*

The proof uses Morse theory and the description of critical points of area on isoperimetric n-gons given in [2]. From the results of [2] and Theorem 1 follows that all critical points of perimeter on area levels F_s have even Morse indices. So the Poincare polynomial of a non-zero area level is lacunary. Hence Morse inequalities imply equalities between Betty numbers and numbers of critical points, which yields the result.

These results can be considerably explicated in the case of convex and conical polygons. Namely, it turns out that the regular n-star is the unique critical point in this case and by duality it is a non-degenerate minimum. It follows that the intersections of area levels with those subspaces are contractible. For $n=3$ and $n=4$, the contracting homotopy can be given explicitly. Moreover, for $n=3$ the leaves of of area folation are smooth non-compact convex surfaces in the space of triangles parameterized by the lengths of the sides. The case of triangle spaces will be discussed in some detail in a forthcoming joint paper of the author with G.Giorgadze.

It is clear that the same considerations are applicable to any pair of functions on polygon space. For example, one can consider the pair consisting of perimeter and Coulomb energy of equal charges placed at the vertices of a polygon. Using results on the Morse indices of Coulomb energy at regular n-stars given in [1] one can obtain an analog of Theorem 3 for the leaves of Coulomb foliation. We postpone a detailed discussion of results of such type since they require separate presentation.

As another illustration of our approach we present an analog of Theorem 1 for cyclic polygons. Recall that the isoperimetric problem for the oriented area of polygonal linkages was discussed in some detail in [6]. The following result can be derived from [2] and [6] by an argument used in the proof of Theorem 1.

Theorem 4. *For any $n \geq 5$ and $a > 0$, the critical points of perimeter on the space of $a-$isochoric cyclic n-gons with fixed circumcircle are regular n-stars. The absolute minimum is attained at convex regular n-gon with area A and the absolute maximum is attained at the regular star polygon with the maximal winding index equal to $[(n-1)/2]$ and oriented area a.*

Information on Morse indices of all regular n-stars can now be derived from [2] and [4] using the isoperimetric duality and Theorem 2. In conclusion we wish to emphasize the approach and results presented in this paper can serve as a paradigm for investigating many extremal problems in polygon spaces.

მათემატიკა

იზოპერიმეტრული ორადობა მრავალკუთხედების სივრცეებში

გ. ხიმშიაშვილი

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(წარმოდგენილია აკადემიის წევრის რ. გამყრელიძის მიერ)

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