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***Mathematics***

# On Area Foliation in Spaces of Triangles

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**We present several results on the geometry of the sets of triangles with fixed area and with fixed area and perimeter. The main result states that the set of triangles with fixed area is contractible and can be represented as a convex surface in the space of triangles. It is also proven that the set of triangles with fixed area and perimeter is represented as a convex closed curve in the space of triangles. The main result is verified by computing the gradient index of Heron polynomial and compared with the results on the area foliation in triangle spaces given in a recent paper by A.Alaoui and A. Zeggar. In conclusion we present a few remarks on the area levels for cyclic quadrilaterals with fixed circumcircle.** 

Polygon space, perimeter, oriented area, area foliation, Heron polynomial, critical point, cyclic quadrilateral, Brahmagupta polynomial

In a recent paper [1], the authors investigate two natural foliations in the spaces of polygons and present some applications of their results in the spirit of approach suggested in [2] (cf. also [3]). In the present paper, we explicate and complement some results of [1] concerned with the area foliation discussed in [1]. More precisely, we show that the non-degenerate leaves of area foliation in the space of triangles are convex non-compact surfaces in the three-dimensional subset of parameters given by three lengths of the sides (Theorem 1). This implies that all those leaves are contractible and yield a construction of contracting homotopy in terms of the gradient flow of perimeter (Theorem 2). These results are complemented and verified by computing the gradient index of Heron polynomial and the Euler characteristic of a non-degenerate leaf of the area foliation for triangles (Proposition 2). As an illustration, we give a visual interpretation of Theorem 1 in the context of Kendall shape space of triangles. In conclusion we outline some possible generalizations and research perspectives suggested by our results.

We begin with presenting the necessary definitions and constructions from [1] and [3]. A non-degenerate polygon in $\mathbb{R}^2$ is an element $\Xi = (M_1, ..., M_n)$ of $\mathbb{R}^{2n}$ such that: (i) for $i \neq j$ the point $M_i$ is distinct from $M_j$; (ii) for any point $k \in \{1, ..., n\}$, the oriented angle $\hat{M}_k = (\overrightarrow{M_k M_{k+1}}, \overrightarrow{M_k M_{k-1}})$ has its measure in $]0, 2\pi[ \setminus \{\pi\}$.

The set of all $n$-gons in the plane $\mathbb{R}^2$ will be denoted $\tilde{\Sigma}_n$. The orbits of the group of the affine isometries of $\mathbb{R}^2$ acting on $\tilde{\Sigma}_n$ are called *geometric polygons* of $\mathbb{R}^2$ and the set of orbits $\Sigma_n$ is $\Xi = (M_1,...,M_n)$ in $\mathbb{R}^2$ is convex if for any $k \in \{1,...,n\}$, the oriented angle $\hat{M}_k = (\overrightarrow{M_k M_{k+1}}, \overrightarrow{M_k M_{k-1}})$ has its measure in $]0,\pi[$. It is called *star-shaped polygon* with respect to the vertex $M$ if for any vertex $N \in \{M_1,...,M_n\}$ the open segment $]M,N[$ is contained in the interior of $\Xi$. A geometric polygon of in $\mathbb{R}^2$ is said to be equilateral if it admits a representative which is equilateral.

Let us define the real-valued functions $p(\omega)$, perimeter, and $a(\omega)$, area, for any geometric polygon $\omega$. For given positive real numbers $r$, $s$, by $F_r = p^{-1}(r)$ and $G_s = a^{-1}(s)$ we denote the level sets of the perimeter and area functions. The collections of these sets are called the perimeter and area foliations, respectively.

We will also use the concept of Kendall shape space defined as follows [4]. Denote by $C^*(k,m)$ the set of all configurations of $k$ labelled points $x_i$ in $\mathbb{R}^m$, such that not all of them coincide, and endow it with the natural topology inherited from in $\mathbb{R}^m$. Let us denote by *Sim(m)* the group of similarities of in $\mathbb{R}^m$ generated by the parallel shifts, rotations and homotheties. This group has an obvious diagonal action on $C^*(k,m)$ and the *Kendall shape space* $K(k,m)$ is defined as the factor-space of $C^*(k,m)$ over this action of *Sim(m)*. As is well known, $K(k,m)$ is a compact and connected Hausdorff topological space which has a natural distance function generating the same topology [4].

In this paper we only need the *planar shape space* $K(k,2)$ and think of its elements as shapes of oriented $k$-gons in the plane. As was shown in [4], the space $K(k,2)$ is homeomorphic to the complex projective space $\mathbb{C}P^{k-2}$. In particular, for $k=3$ the shape space of labelled triangles $K(3,2)$, is isometric to a sphere of radius $1/2$ in $\mathbb{R}^3$ endowed with the geodesic (great circle) distance [4]. This sphere contains several distinguished subsets corresponding to specific triangular shapes such as regular (poles), aligned (equator) and degenerate (3 points on equator) [4]. In the sequel we interpret our main result in terms of Kendall sphere.

One of the classical results on isoperimetric problem states that the regular triangle has the maximal area among all triangles with fixed perimeter [1]. In other words, it can be interpreted as a critical point of area considered as a differentiable function on the space of triangles with fixed perimeter endowed with a natural smooth structure. As usual one can also consider the dual extremal problem which in our case means minimization of perimeter of triangles with fixed oriented area. To this end one needs to understand the geometry and topology of the area foliation of triangles which is our main concern in the sequel. A general approach to dual isoperimetric problem for planar $n$-gons is developed in [5].

We are now ready to give precise formulations and rigorous proofs of our results.

**Theorem 1.** *All leaves of area foliation in $G_3$ are smooth convex non-compact two-dimensional surfaces in three-dimensional open octant $\mathbb{R}^3_+$.*

**Proof.** Obviously, each leaf is a level set of the Heron polynomial

$$H(x,y,z) = (x+y+z)(-x+y+z)(x-y+z)(x+y-z) \tag{1}$$

As was proven in [1] the area level $\{H = a\}$ is a smooth two-dimensional surface in $G_3$. We slice such a leaf by the planes $\{x+y+z = const\}$ and show that those intersections are convex closed curves encircling the class of regular triangle on this level. In other words, we consider the intersection of the level set $\{H = a\}$ with the leaf of perimeter foliation $\{p = 2c\}$. This means that we can substitute $z = 2c - x - y$ in (1) and consider the corresponding closed curve in the plane $\{x+y+z = 2c\}$. Convexity is invariant under orthogonal projection so it is sufficient to set one coordinate equal to zero, say $\{z = 0\}$, and show that the projected curve in the $xy$-plane is convex. Clearly, the projected curve is defined by the equation $\{f(x, y) = d\}$, where $d = 2c$ and

$$f(x, y) = c(2c - x)(2c - y)(x + y). \tag{2}$$

We now compute the curvature of this implicit curve by the well-known formula:

$$k = \frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{\left(f_x^2 + f_y^2\right)^{3/2}}. \tag{3}$$

For our purposes it is sufficient to study the sign of the numerator of (3). After a rather long but completely straightforward computation, possible both with or without use of computer , one finds all partial derivatives up to second order. Namely,

$$\begin{cases} f_x = -c(2c - y)(x + y) + c(2c - x)(2c - y), & f_{xx} = -2c(2c - y), \\ f_y = -c(2c - x)(x + y) + c(2c - x)(2c - y), & f_{yy} = -2c(2c - x), \end{cases} \tag{4}$$

$$f_{xy} = c(x + y) + c(2c - y) - c(2c - x).$$

It follows that the numerator of (3) is equal to:

$$\begin{gathered} -32c^7x - 32c^7y + 32c^6x^2 + 128c^6xy + 96c^6y^2 - 96c^5x^2y - 176c^5xy^2 - 80c^5y^3 \\ -8c^4x^4 + 16c^4x^3y + 88c^6x^2 + 128c^4x^2y^2 + 80c^4xy^3 + 16c^4y^4 + 2c^3x^5 + 2c^3x^4y \\ -12c^3x^3y^2 - 20c^3x^2y^3 - 8c^3xy^4. \end{gathered} \tag{5}$$

Notice now that on the projected leaf one has the equality $x + y = 2c$. Inserting $y = 2c - x$ in (4) we are left with the univariate polynomial depending on parameter c and having the form:

$$-64c^8 + 128c^7x - 96c^6x^2 + 32c^5x^3 - 4c^4x^4. \tag{6}$$

To show convexity of (3) it is sufficient to show that this polynomial does not change its sign in the interval [0, 2c]. This of course would follow if we show that (6) has no real roots in this interval. To do this one can compute its discriminant and investigate distribution of its real roots. A standard application of Sturm's algorithm shows that, indeed, there are no real roots of (6) in the segment [0, 2c] so the result follows. The computations involved in Sturm algorithm are routine and therefore omitted.

**Corollary 1.** All leaves of area foliation in $\mathbb{R}^3_+$ are contractible.

Moreover, since we know that the regular triangle is the unique critical point in each leaf it follows that the gradient descent along the gradient lines of perimeter transports each point to the single minimum point.

**Corollary 2.** Each intersection of a leaf of area fibration with the sublevel set of perimeter is contractible.

Since it is known that the class of regular triangle is the unique critical point of perimeter on a leaf of area foliation, the standard arguments of Morse theory yield the following result.

**Theorem 2.** The negative gradient flow of perimeter on a leaf of area foliation of triangle space carries each point to the class of a regular triangle in finite time.

This result can be made explicit by computing the gradient flow of perimeter on a leaf of area foliation, which in fact reduces to a rather elementary geometric problem: given a non-regular triangle construct its first order deformation, i.e. vectors at its vertices, such that infinitesimal shift along this vectors leaves area invariant but decreases the perimeter. Such construction is geometrically, i.e. for an *individual* triangle, rather obvious but its invariant description, i.e. in terms of *geometric* triangles, requires the definitions and preparations presented above.

To obtain explicit formulas for the gradient of perimeter restricted to a leaf of area foliation one has to compute the projection of the gradient of perimeter in the parameter space on the tangent space to the leaf considered. The gradient of perimeter in the parameter space is the constant vector (1, 1, 1) and the projection to the tangent plane of leaf can be computed by taking the gradient minus its projection on the unit normal to the leaf. The normal vector is given by the normalized vector product of the partial derivatives given by the formulas (4) above. The negative of this vector field gives exact formulas for the gradient descent flow.

**Remark 1.** To visualize this result let us fix one vertex at the origin and place another one on the real axis. Then the gradient of perimeter is represented by two vectors at the second and third vertex giving a first order deformation which preserves the oriented area and decreases the perimeter. It is easy to calculate these vectors in the case of isosceles triangle and it would be interesting to compute them for general (scalene) triangle.

**Remark 2.** One can verity these results by computing the Euler characteristic of leaves of area foliation. To this end one computes first the local topological degree of Heron polynomial, which can be done using the signature formula for the local topological degree given in [6]. After that one computes the Euler characteristic of local level surface using another formula from [6], which appears to be equal to one. It remains to notice that this gives the Euler characteristic of the whole leaf of area foliation due to homogeneity of Heron polynomial, which confirms the contractibility of leaves of area foliation.

Our results can be interpreted in terms of the Kendall sphere $S_K$. As follows form the very definition of Kendall sphere one can take the set of (unordered) angles $u, v, w$ as local coordinates on $S_K$. In other words, one considers triples of positive numbers $u, v, w$ with the sum equal to $2\pi$ and interpretes them as angles of a triangle. Consider now the image of the set of triangles with fixed area $s$ and perimeter $p$ in the space of angle coordinates. It is obviously a curve $Z(p,s)$ in the triangle orthogonal to the bisector of the first octant. It is easy to verify that equation of this curve in the space of angle coordinates is given by the two equations

$$u+v+w=2\pi, \quad \frac{(\sin u+\sin v+\sin w)^2}{\sin u \sin v \sin w}=\frac{a^2}{2s}.$$

One can compute the curvature of this curve in the same way as was done in the proof of Theorem 1 and verify that it is everywhere positive, which gives the following conclusion.

**Corollary 3.** For all positive p, s, the set Z(p,s) is a smooth convex curve.

Taking the image of curve $Z(p,s)$ in the Kendall sphere one also comes to the following conclusion.

**Corollary 4.** The shapes of triangles with fixed positive area and perimeter form a smooth convex curve in the upper hemisphere of the Kendall sphere.

Such an interpretation suggests certain new possibilities. For example, for given p, s, one can try to compute or estimate the length of such curves in the metric of the Kendall sphere. We omit discussion on this specific problem here and proceed by describing a more general way of extending our results.

To this end we notice that the idea of considering a pair of dual foliations can be applied to other pairs of functions on triangle space. This leads to interesting results if one takes Coulomb energy of equal charges placed at the vertices of triangle. More precisely, denoting by $x, y, z$ the sidelengths of geometric triangle we put $E(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ and consider the foliation of the first octant by the level surfaces of Coulomb energy $Y_b = \{E(x,y,z) = b\}$.

**Theorem 3.** All leaves of Coulomb foliation in the first octant are smooth convex non-compact two-dimensional surfaces.

The proof is analogous to the proof of Theorem 1 and we also have similar corollaries.

**Corollary 5.** All leaves of Coulomb foliation in the first octant are contractible.

Moreover, since we know that the regular triangle is the unique critical point in each leaf it follows that the gradient descent along the gradient lines of perimeter transports each point to the single minimum point.

**Corollary 6.** Each intersection of a leaf of Coulomb foliation with the sublevel set of perimeter is contractible.

Since it is known that the class of regular triangle is the unique critical point of perimeter on a leaf of area foliation, the standard arguments of Morse theory yield the following result.

**Theorem 4.** The negative gradient flow of perimeter on a leaf of Coulomb foliation of triangle space carries each point to the class of a regular triangle in finite time.

**Remark 3.** Coulomb energy of equally charged vertices can be used to introduce new coordinates in the triangle space which are given by symmetric functions of sidelengths. Namely, our next result shows that one can take perimeter, area and Coulomb energy as global coordinates on the triangle space.

**Theorem 5.** The values of perimeter, area and Coulomb energy of a geometric triangle uniquely define the sides of triangle.

**Proof.** To prove this let us first show that the following system of equations for unknown sidelengths $x, y, z$ can have not more than one solution up to the order of unknowns:

$$x + y + z = a, \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = b, \quad H(x,y,z) = c, \tag{7}$$

where $H(x,y,z) = (x+y-z)(x-y+z)(-x+y+z)(-x-y-z)$.

To solve this system we introduce the elementary symmetric functions

$$\sigma_1(x,y,z) = x+y+z, \quad \sigma_2(x,y,z) = xy+xz+yz, \quad \sigma_3(x,y,z) = xyz,$$

rewrite first two equations from (7) as

$$\sigma_1(x,y,z)=a,\ \ yz+xz+xy=bxyz,\ \ \sigma_2(x,y,z)=b\sigma_3(x,y,z)$$

and express the Heron polynomial as follows:

$$H(x,y,z)=(x+y-z)(x-y+z)(-x+y+z)(-x-y-z)=$$
$$(x+y+z)^4-4(xy+xz+yz)(x+t+z)^2+8(x+y+z)xyz=$$
$$\sigma_1^4(x,y,z)-4\sigma_2(x,y,z)\sigma_1^2(x,y,z)+8\sigma_1(x,y,z)\sigma_3(x,y,z).$$

It follows that

$$a^4-4a^2b\sigma_3(x,y,z)+8a\sigma_3(x,y,z)=c$$

and we obtain

$$\sigma_3(x,y,z)=\frac{c-a^4}{8a-4a^2b},\ \ \sigma_2(x,y,z)=\frac{b(c-a^4)}{8a-4a^2b}.$$

By Vieta theorem, having found the elementary symmetric functions of sidelengths we can find the sidelengths themselves by solving the cubic equation:

$$\xi^3-\sigma_1\xi^2+\sigma_2\xi-\sigma_3=0.$$

This already shows that there can exist not more than one unordered triple of real numbers consisting of roots of this equation. If the triple *a, b, c* were the perimeter, area and Coulomb energy of a certain triangle then our procedure will uniquely restore the triple of sidelengths, as was claimed. The proof is complete.

**Remark 4.** Analyzing the discriminant of the aforementioned cubic equation one can indicate conditions on *a, b, c* which guarantee existence of three positive roots (some of which may coincide) and show that they will satisfy triangle inequalities. In this way one can explicitly describe the domain of values for the coordinates introduced above.

Similar constructions and results can be developed for cyclic quadrilaterals using the Brahmagupta formula [7]. In particular the leaves of area foliation are also contractible in this case. We do not discuss the case of quadrilaterals for the reason of space.

It is of course natural to investigate possible analogs of our results for $n$-gons with $n > 4$. It should be at once noted that a direct generalization of our considerations is impossible due to absence of analogs of Heron and Brahmagupta formulas. Moreover, as was explained in [5] non-degenerate leaves of area foliation are not contractible for $n > 4$. Actually, the same should be true for Coulomb foliation. So an interesting problem is to investigate the topology of those leaves, for example, to calculate their homology groups. Further development of this topic for arbitrary $n$ requires more sophisticated approaches and methods, for which the results of present paper may serve as a paradigm.

*მათემატიკა*

# სამკუთხედების სივრცეების ფართობის ფიბრაციის შესახებ

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მოყვანილია რამდენიმე შედეგი სამკუთხედების ფართობის ფიბრაციის შესახებ. კერძოდ, დამტკიცებულია, რომ ამ ფიბრაციის ფენები ამოზნექილი ზედაპირებია პარამეტრთა სამგანზომილებიან სიცრცეში. დამტკიცებულია აგრეთვე, რომ ყველა ფენა მოჭიმვადია. გამოთვლილია ჰერონის მრავალწევრის გრადიენტის ინდექსი და ნაჩვენებია, რომ ამ შედეგებიდან გამომდინარეობს ა. ალაუის და ა.ზეგარის სტატიის ძირითადი შედეგი.

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