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Mathematics

On a Fair Price of the European Option

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(Presented by Academy Member Elizbar Nadaraya)

In the paper one discrete model of financial market with k bonds and one stock is considered. Unified interest rate of the bonds is introduced. Explicit forms of the fair price and minimal hedge are obtained for the European contingent claim. *© 2020 Bull. Georg. Natl. Acad. Sci.*

Financial market, fair price, European option, minimal hedge

Let us consider binomial financial (B,S) – market driven by the following recurrent formulas:

$$B_n = (1+r_n)B_{n-1}, \tag{1}$$

$$S_n = (1+\rho_n)S_{n-1}, \tag{2}$$

where r_n is the interest rate and ρ_n is the sequence of independent identically distributed random variables. ρ_n takes only two possible values a and b, $a < b$, with probabilities $1-p$ and p $(0 < p < 1)$ respectively [1, 2]. It is assumed that bank account or bond B is risk free, while stock S is risk asset.

The following is true

Lemma. Suppose, that B_n presents sum of k number of bonds

$$B_n = \sum_{i=1}^{k} B_n^{(i)} = \sum_{i=1}^{k} (1+r^{(i)})B_{n-1}^{(i)}, \tag{3}$$

where $r^{(i)}, i = 1,...,k,$ are constant interest rates. Then it follows:

$$r_n \frac{\sum_{i=1}^{k} r^{(i)} B_{n-1}^{(i)}}{\sum_{i=1}^{k} B_{n-1}^{(i)}}. \tag{4}$$

Proof. We have:

$$(1+r_n)B_{n-1} = \sum_{i=1}^{k} (1+r^{(i)})B_{n-1}^{(i)},$$

$$(1+r_n)\sum_{i=1}^{k} B_{n-1}^{(i)} = \sum_{i=1}^{k} (1+r^{(i)}) B_{n-1}^{(i)},$$

and equality (4) follows immediately.

Note that using equality (4), the so called risk-neutral probability measure can be constructed [4]:

$$p^* = \frac{r_n - a}{b - a}.$$

Remark also that the following equality is valid

$$E^* \rho_n = r_n,$$

where E^* denotes expectation by the measure p^*.

Suppose now, that European option with pay-off function f_N is given, as, for instance, standard call option $f_N = (S_N - K)^+$ [3, 4].

Seller of the option has commitment to the customer to pay amount f_N at the terminal moment N. For that seller uses cost of the option and trades with bonds and stocks in the way that guarantees amount f_N at the moment N.

One of the main problems is to determine minimal price of the option that gives possibility to complete contingent claim f_N .

Below we assume that filtered probability space $(\Omega, \Im, \Im_n, P)$ is given, where $\Im_n = \sigma\{S_0,...,S_n\}$ is minimal σ -algebra subtended by $S_0,...,S_n$.

Let $X_0 = x > 0$, be the initial amount of investor and at the moment n the capital related to the portfolio $\pi_n = (\beta_n, \gamma_n)$ is:

$$X_n^\pi = \beta_n B_n + \gamma_n S_n,$$

where β_n and γ_n are quantities of bonds and stocks.

Portfolio $\pi_n = (\beta_n, \gamma_n)$ is called (x, f, N) hedge, if

$$X_0^\pi = x,$$

$$X_N^\pi = \beta_N B_N + \gamma_N S_N \geq f_N.$$

If $X_N^\pi = f_N$, then π is minimal hedge.

The set of (x, f, N) hedges we denote by $\Pi(x, f, N)$. Expression

$$C(N) = \inf\{x > 0 : \Pi(x, f, N) \neq \varnothing\}$$

is called fair price of the option.

Theorem 1. Let financial market (B, S) be given by (1), (2) recurrent equalities. Then for the European contingent claim with payoff f_N the fair price is:

$$C(N) = \varepsilon_N^{-1}(U) E^* f, \tag{5}$$

where E^* is the expectation under martingale measure P^* and

$$\varepsilon_n(U) = \prod_{k=1}^{n} (1 - \Delta U_k),$$

$$U_n = \sum_{k=1}^{n} r_k.$$

Proof. Let $\pi \in \Pi(x, f, N)$. The market defined by (1), (2) is complete and arbitrage free. It is known [3] that sequences

$$M_n^\pi = X_n^\pi B_n^{-1} = X_0 B_0^{-1} + \sum_{k=1}^{n} B_k^{-1} \gamma_k S_{k-1}(\rho_k - r_k) =$$

$$X_0 B_0^{-1} + \sum_{k=1}^{n} \tilde{\gamma}_k (\rho_k - r_k),$$

where $\tilde{\gamma}_n = \dfrac{\gamma_n S_{n-1}}{B_n}$ and $m_n = \sum_{k=0}^{n} (\rho_k - r_k)$ represent martingales with respect to measure P^*.

Then we have:

$$x = X_0 = E^* \varepsilon_N^{-1}(U) X_N^\pi = \varepsilon_N^{-1}(U) E^* X_N^\pi,$$

and it follows that $x \geq \varepsilon_N^{-1}(U) E^* f$.

If π is minimal (x, f, N) hedge, then

$$x = \varepsilon_N^{-1}(U) E^* f = C(N),$$

and thus (5) is proved.

Theorem 2. If the financial (B, S) market is given by (1), (2) recurrent formulas, then:

a) There exists minimal hedge $\pi^* = (\beta_n^*, \gamma_n^*), n \leq N$, and

$$\gamma_n^* = \frac{\tilde{\gamma}_n B_n}{S_{n-1}}, \tag{6}$$

$$\beta_n^* = \frac{X_{n-1}^{\pi^*} - \gamma_n^* S_{n-1}}{B_{n-1}}, \tag{7}$$

where $(\tilde{\gamma}_n), n \leq N$, is the predictable sequence.

b) The capital corresponding to minimal hedge is

$$X_n^{\pi^*} = E^*(\varepsilon_N^{-1}(U) \varepsilon_n(U) f / \Im_n).$$

Proof. Denote:

$$\gamma_k^* = \tilde{\gamma}_k B_k S_{k-1}^{-1}, k = 1, ..., N.$$

Then, considering [3] we have:

$$M_n^* = E^*(B_N^{-1} f / \Im_n) = \frac{x}{B_0} + \sum_{k=1}^{n} \frac{\gamma_k^* S_{k-1}}{B_k} \Delta(\rho_k - r_k).$$

It is known from [3] that there exists self-financing strategy $\pi^* = (\beta_n^*, \gamma_n^*)$, for which the discounted capital process $X_n^{\pi^*} / B_n$ equals to $M_n^*, n = 0, 1, ..., N$. Indeed:

$$\gamma_1^* = \frac{\tilde{\gamma}_1 B_1}{S_0},$$

and we define β_1^* by the equality

$$\beta_1^* = \frac{x - \gamma_1^* S_0}{B_0}.$$

Then using simple transformation we get:

$$M_1^{\pi^*} = \beta_1^* + \gamma_1^* \frac{S_1}{B_1} = \frac{x}{B_0} + \gamma_1^* \left(\frac{S_1}{B_1} - \frac{S_0}{B_0} \right) =$$

$$\frac{x}{B_0} + \frac{\tilde{\gamma}_1 B_1}{S_0} \left(\frac{S_1}{B_1} - \frac{S_0}{B_0} \right) = \frac{x}{B_0} + \tilde{\gamma}_1 \left(\frac{S_1}{S_0} - \frac{B_1}{B_0} \right) =$$

$$\frac{x}{B_0} + \tilde{\gamma}_1 (\rho_1 - r_1) = M_1^*.$$

By induction principle it is easy to see that the strategy $\pi^* = (\beta_n^*, \gamma_n^*)$, determined by (6), (7), satisfies the following equalities:

$$\frac{X_n^{\pi^*}}{B_n} = M_n^{\pi^*} = M_n^* = E^*(B_N^{-1} f / \Im_n),$$

$$X_n^{\pi^*} = E(\varepsilon_N^{-1}(U) \varepsilon_n(U) f / \Im_n),$$

$$x = X_0^{\pi^*} = \varepsilon_N^{-1}(U) E^*(f / \Im_n),$$

$$X_n^{\pi^*} = f.$$

and thus the Theorem is proved.

მათემატიკა

ევროპული ოფციონის სამართლიანი ფასის შესახებ

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