საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 14, №2, 2020
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 14, no. 2, 2020

Physics

Relativistic Equations and Quark Confinement Problem

Anzor Khelashvili[*] and Teimuraz Nadareishvili[**]

[*] *Academy Member, Institute of High Energy Physics, Ivane Javakhishvili Tbilisi State University, Tbilisi, Georgia*
[**] *Institute of High Energy Physics, Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University, Tbilisi, Georgia*

The Dirac equation in an external field with infinitely rising central potential, which is a fourth component of the Lorentz-vector, has only continuous spectrum owing to a leakage into infinite barrier (the Klein paradox). The same situation happens in the Breit equation. It is known that the inclusion of the Lorentz-scalar potential changes this undesirable property. From theoretical point of view vector interaction is preferable, because the color gluons, by exchange of which the strong force between quarks is generated, are vector particles. On the other hand, all relativistic equations for both potentials are reduced to the same Schrodinger equation, in the framework of which quarkonium (bound states of quark and antiquark) spectrum is described theoretically rather well. For a long time it was unclear which relativistic equation will be more suitable for quarkonium problems. Below we consider the most general quasipotential equation for spin-1/2 quarks, which was derived by one of the authors (A.K.). This equation involves full information of the quantum field theory and is 3-dimensional. In particular cases it follows all the known three dimensional equations. The quasipotential contains projective operators in case of instantaneous kernels and reduces to the Salpeter equation. These projective operators in a reasonable approximation induce scalar potential from the vector one. Therefore, the Klein paradox is avoided. As a result, bound state solutions appear. At last, the obtained earlier radial equation is exhibited and the confinement phenomenon is demonstrated.

Klein paradox, Dirac equation, scalar potential

It is well known that the Dirac equation in central symmetric field, which is a zero component of the Lorentz four-vector and is infinitely increasing, has a pathological property – the leakage through the infinite wall occurs or equivalently, the wave function has an oscillating (i.e. non-normalizable) asymptotic behavior. This unusual property is known as the Klein Paradox [1]. In case of quarks and antiquarks this problem is usually avoided by introducing additional Lorentz-scalar interaction [2,3]. Particular importance is given to the equal mixture of scalar and vector potentials, which excludes the spin-orbital coupling in Dirac equation [2-4].

It seems quite natural to ask for relativistic covariant QFT equations for bound states such as the Bethe-Salpeter (BS) equation. There is an opinion that only vector exchange interaction is not desirable here due to the Klein paradox. At the same time, it is clear that if the Klein paradox exists here, the situation would be rather vague because the fundamental particles that transfer interaction between quarks are vector gluons and they must provide confinement of quarks into hadrons. Otherwise all the attempts to obtain singular (i.e. $1/q^4$) behavior of gluon propagator [5] in the infrared area would be useless.

Last years a lot of articles have been dedicated to 3-dimensional relativistic equations which follow from BS equation by using various reduction methods. These equations are interesting because they give the possibility to study quarkonium spectra for different potentials.

Despite the fact that 3-dimensional relativistic equations have a long history of study, in recent years much attention has been paid to the formulations in which by infinite overweighting of one of constituent particles the problem is reduced to the Dirac equation for light particle in an external field.

This paradigm is not clear for us, because, in our opinion, the information on second quantization could not be lost completely whereas one of the particles becomes infinitely heavier. Really when we use 3-dimensional (for example, instantaneous) kernel in BS equation the latter is reduced to Salpeter equation which differs from two-particle Dirac (Breit) equation by projection operators in interaction term both only on positive and negative frequencies, separately, but not contain their interference. These operators are the only remnants of secondary quantization and they cannot be neglected.

It was supposed [4,6] that QFT equations would be free from the Klein paradox. There are remarkable works [7,8] dedicated to the study of Salpeter equation from this point of view. However final conclusions are made basing on some approximations. Below we consider the problem on theoretical grounds of the most general quasipotential approach [9] for spin-1/2 particles [10], which is closer to the BS equation.

For completeness we include here a short derivation of quasipotential equation in the instant form. The critical quantity is the two-time Green function which in momentum space is defined as an integral over relative energy (in CM frame, $\boldsymbol{P}=0$)

$$\tilde{G}(E;\boldsymbol{p},\boldsymbol{q})=\int_{-\infty}^{\infty} dp_0 dq_0 G(E;\boldsymbol{p},p_0;\boldsymbol{q},q_0). \tag{1}$$

The equation for one time BS amplitude (wave function) follows [9] from the obvious relation

$$\tilde{G}^{-1}\tilde{G}=1 \tag{2}$$

near the poles of bound state.

Here $\tilde{G}^{-1}$ may be constructed by inverting the integrated BS equation:

$$\tilde{G}=\tilde{G}_0+\widetilde{G_0 K G}. \tag{3}$$

Considering (2) near the bound state poles one can derive the three-dimensional equation:

$$\tilde{G}^{-1}\Psi(\boldsymbol{p})=0, \qquad E\approx E_B, \tag{4}$$

where $\Psi(\boldsymbol{p})$ is the instantaneous BS amplitude

$$\Psi(\boldsymbol{p})=\int_{-\infty}^{\infty} dp_0 \chi(\boldsymbol{p},p_0). \tag{5}$$

Hence, the construction of three-dimensional equations is related to the inversion of $\tilde{G}$, which, according to (2) and (3), requires to invert the two-time free Green function $\tilde{G}_0$. In the spin less case it can be obtained easily [9]. But for fermions $\tilde{G}_0$ is a singular matrix, since [10]

$$\tilde{G}_0=-2\pi i\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\left[E-H_1(\boldsymbol{p})-H_2(-\boldsymbol{p})\right]^{-1}\beta_1\beta_2\Pi(-\boldsymbol{p}). \tag{6}$$

Here Π is the Salpeter projective operator

$$\Pi(\boldsymbol{p}) = \Lambda^{(+)}{}_1(\boldsymbol{p})\Lambda^{(+)}{}_2(-\boldsymbol{p}) - \Lambda^{(-)}{}_1(\boldsymbol{p})\Lambda^{(-)}{}_2(-\boldsymbol{p}). \tag{7}$$

People avoided this trouble mainly by projecting the original equation (3) onto the subspace of positive frequencies [11]. It is clear that the full function $\tilde{G}$ does not possess projective properties. Therefore, there arises a natural question: How to obtain an equation for the full three-dimensional wave function? In the past broad use was made of the Breit equation [12] – an analogous of the Dirac equation in the hydrogen problem. This equation was written for the full three-dimensional wave function. Can one find any ground for this equation in the quantum field theory?

One of the possible solutions was suggested in [10]. It consists in the consideration of auxiliary function:

$$\tilde{G}' = F(\boldsymbol{p}) + \widetilde{G_0 \cdot K \cdot G} = \tilde{G} + F(\boldsymbol{p})\big(1 - \Pi(-\boldsymbol{p})\big), \tag{8}$$

where

$$F(\boldsymbol{p}) = -2\pi i\left[E - H_1(\boldsymbol{p}) - H_2(-\boldsymbol{p})\right]^{-1}\beta_1\beta_2. \tag{9}$$

Near the bound state poles $\tilde{G}'$ coincides with $\tilde{G}$. Therefore, one can postulate the following equation for the bound states

$$\tilde{G}'^{-1}\Psi_E = 0. \tag{10}$$

Since $\tilde{G}'^{-1}\tilde{G}' = 1, \quad , E \approx E_B$. At the same time

$$\tilde{G}'^{-1} = \left[1 - F^{-1}\cdot\widetilde{G_0KG}\right]^{-1}\cdot F^{-1}. \tag{11}$$

Therefore, we obtain the desired equation

$$\left[E - H_1(\boldsymbol{p}) - H_2(-\boldsymbol{p})\right]\Psi_E(\boldsymbol{p}) = \int d\boldsymbol{q}\hat{V}(E;\boldsymbol{p},\boldsymbol{q})\Psi_E(\boldsymbol{q}), \tag{12}$$

in which the quasipotential is defined formally as follows:

$$\hat{V} = \beta_1\beta_2F^{-1}\cdot\widetilde{G_0KG}G'^{-1}. \tag{13}$$

In the instantaneous approximation $(K = K_{st})$ the Eq. (12) is reduced to the Salpeter equation. In fact,

$$\hat{V}\Psi_E = \beta_1\beta_2F^{-1}\widetilde{G_0}K_{st}\tilde{G}\cdot\tilde{G}'^{-1}\Psi_E = \beta_1\beta_2F^{-1}F\Pi K_{st}\tilde{G}\cdot\tilde{G}'^{-1}\Psi_E = \beta_1\beta_2\Pi K_{st}\Psi_E, \tag{14}$$

because in the bound state poles $\tilde{G}' = \tilde{G}$.

Using (14) we can rewrite our quasipotential equation (12) in the form of the Salpeter one

$$\left[E - H_A(\boldsymbol{p}) - H_B(-\boldsymbol{p})\right]\Phi(\boldsymbol{p}) = \Pi(\boldsymbol{p})\int d\boldsymbol{q}\hat{V}(\boldsymbol{p}-\boldsymbol{q})\Phi(\boldsymbol{q}). \tag{15}$$

In general, the construction of a quasipotential is a very tedious problem, especially in QCD at large distances, where the perturbative expansions lose any sense. Therefore, we are forced to consider phenomenological quasipotentials, that are permitted by general logic of this method [9]. The remined part will be devoted to the phenomenological equation:

$$\left[E - H_1(-i\nabla) - H_2(i\nabla)\right]\Psi(\boldsymbol{r}) = \hat{V}(r)\Psi(\boldsymbol{r}), \tag{16}$$

to which our equation (12) is reduced in coordinate space for local quasipotentials. At the first stage $\hat{V}(r)$ can be taken independent of the derivatives. The simplest choice $\hat{V}(r) = V(r)$ corresponds to the exchange of the 4th component of the Lorentz-vector. Passing to the radial equations and considering the limit $r \to \infty$, one can find that the radial functions do not decrease but oscillate (Klein paradox), i.e. there are no bound states. The Klein paradox can be avoided if we use relevant matrix structures, following from projective operators above. In the large distance limit or small linear momentum extensively used projective operators behave as

$$\Lambda_1^{(+)}(p)\Lambda_2^{(+)}(-p)-\Lambda_1^{(-)}(p)\Lambda_2^{(-)}(-p) \;\to\; \frac{1}{2}(\beta_1+\beta_2). \tag{17}$$

In the more general case of non-static kernels, it is possible to have a formulation with projection operator

$$\Lambda_1^{(+)}(\boldsymbol{p})\Lambda_2^{(+)}(-\boldsymbol{p})+\Lambda_1^{(-)}(\boldsymbol{p})\Lambda_2^{(-)}(-\boldsymbol{p}) \to \frac{1}{2}(1+\beta_1\beta_2). \tag{18}$$

In the first (17) formulation it is inconvenient to replace this operator by unity, since the Klein paradox occurs. Either one must make a non-relativistic reduction in powers of υ^2/c^2 in the complete equation or use part of this operator, for example, make a restriction to terms of order $0(1)$ after expansion in υ^2/c^2. It is readily to be seen that there remain from the above projection operators pure matrix structures (last forms in (17) and (18), which in Kummer's notation means [13]

$$\left\{\hat{V}\Psi\right\}=\frac{1}{2}V(\beta\Psi+\Psi\beta), \qquad \left\{\hat{V}\Psi\right\}=\frac{1}{2}V(\Psi+\beta\Psi\beta). \tag{19}$$

Unifying these structures, we derive the equation:

$$\begin{aligned}&(-i\boldsymbol{\alpha}\nabla+m_1\beta)\Psi(\boldsymbol{r})+\Psi(\boldsymbol{r})(-i\boldsymbol{\alpha}\nabla+m_2\beta)=E\Psi(\boldsymbol{r})-\\ &-\frac{1}{2}V_1(r)\big(\beta\Psi(\boldsymbol{r})+\Psi(\boldsymbol{r})\beta\big)-\frac{1}{2}V_2(r)\big(\Psi(\boldsymbol{r})+\beta\Psi(\boldsymbol{r})\beta\big).\end{aligned} \tag{20}$$

In such a formulation, it is found that the scalar potential here does not have an independent nature (i.e., does not necessarily admit the existence of scalar gluons) – it can be induced by the vector gluon on the transition from the covariant Bethe-Salpeter equation to the one-time equation of quasipotential type [10].

The angular decomposition can be performed, for example, by the method of [12]. Omitting details, we exhibit below the radial equations for the states with parity $\varepsilon_P=(-1)^{J+1}$, so-called $\rho-\psi$ trajectory):

$$u''+\left\{\frac{M^2-\kappa^2}{4M}\left[M-V_2-\frac{(\mu+V_1)^2}{M-V_2}\right]-\frac{J(J+1)}{r^2}\right\}u=0. \tag{21}$$

Here $\mu=m_1+m_2$, $\kappa=m_1-m_2$ and $M\equiv E$ (CM system). It is easy to see that in separate cases $V_1=0$ or $V_2=0$ the then obtained equations are free of the Klein paradox as $r\to\infty$. The same situations take place for other trajectories [14].

This consideration tells us that the correct relativistic equations contain true information about the variant structure of quasipotential in contrast to pure Dirac equation in an external field. Therefore, when one considers the analogous problems, one must make use of two particle relativistic equations and reduce them by infinite overweighting of one of particles. After all we become to the modified Dirac equation,

$$\left[\boldsymbol{\alpha}\boldsymbol{p}+\beta m+\Lambda^{(+)}(\boldsymbol{p})V(r)\right]\Psi(\boldsymbol{r})=E\Psi(\boldsymbol{r}). \tag{22}$$

It was shown in [14,15], that if this equation is free from the Klein paradox, the same will be true for Salpeter equation and if for the rising potentials the Schrodinger equation has only discrete spectrum, the same conclusion remains for the modified equation (22), (see [16]). Therefore, potential confinement of quarks must be investigated in the frame of true relativistic equations in the framework of quantum field theory.

ფიზიკა

რელატივისტური განტოლებები და კვარკის კონფაინმენტის პრობლემა

ა. ხელაშვილი* და თ. ნადარეიშვილი**

** აკადემიის წევრი, ივანე ჯავახიშვილის თბილისის სახელმწიფო უნივერსიტეტი, მაღალი ენერგიების ფიზიკის ინსტიტუტი, თბილისი, საქართველო*
*** ივანე ჯავახიშვილის თბილისის სახელმწიფო უნივერსიტეტი, მაღალი ენერგიების ფიზიკის ინსტიტუტი, ზუსტ და საბუნებისმეტყველო მეცნიერებათა ფაკულტეტი, თბილისი, საქართველო*

დირაკის განტოლებას გარე ველში, უსასრულოდ ზრდადი ცენტრალური პოტენციალით, რომელიც არის ლორენც-ვექტორის მეოთხე კომპონენტი, აქვს მხოლოდ უწყვეტი სპექტრი, უსასრულო ბარიერში გაჟონვით გამოწვეული (კლეინის პარადოქსი). იგივე სიტუაცია გვხვდება ბრეიტის განტოლებაშიც. ცნობილია, რომ ლორენც-სკალარის ჩართვა ცვლის ამ არასასურველ თვისებას. თეორიული თვალსაზრისით, ვექტორული ურთიერთქმედება არის უფრო მისაღები, რადგან ფერადი გლუონები, რომელთა საშუალებით კვარკებს შორის ძლიერი ძალები გენერირდება, არის ვექტორული ნაწილაკები. მეორე მხრივ, ყველა რელატივისტური განტოლებიდან რედუქციით მიიღება ერთი და იგივე შრედინგერის განტოლება, რომლის ფარგლებში კვარკონიუმები (კვარკის და ანტიკვარკის ბმული მდგომარეობანი) თეორიულად საკმაოდ კარგად აღიწერება. დიდი ხნის განმავლობაში არ იყო ნათელი, თუ რომელი რელატივისტური განტოლება იყო უფრო მოსახერხებელი კვარკონიუმის პრობლემაში. ჩვენ განვიხილავთ ყველაზე ზოგად კვაზიპოტენციალურ განტოლებას ½-სპინიანი ნაწილაკებისათვის, რომელიც ადრე მიღებული იყო ერთ-ერთი ჩვენგანის (ა.ხ.) მიერ. ეს განტოლება შეიცავს ველის კვანტური თეორიის სრულ ინფორმაციას და არის 3-განზომილებიანი. ცალკეულ კერძო შემთხვევებში, აქედან გამომდინარეობს ყველა ცნობილი 3-განზომილებიანი განტოლება. კვაზიპოტენციალი შეიცავს პროექციულ ოპერატორებს მყისიერი გულების შემთხვევაში და მიიყვანება სოლპიტერის განტოლებაზე. გონივრულ მიახლოებაში ეს პროექციული ოპერატორები ინდუცირებს სკალარულ პოტენციალებს ვექტორულიდან. ამიტომ კლეინის პარადოქსი თავიდან აიცილება და წარმოიქმნება ბმული მდგომარეობების ამოხსნები. ბოლოს, ადრე მიღებული რადიალური განტოლებაა ამოწერილი და კონფაინმენტის (კვარკების ჩაჭერის) მოვლენა არის დემონსტრირებული.

REFERENCES

1. Bjorken J.D. and Drell S.D. (1964) Relativistic quantum mechanics, McGraw-Hill, New York.
2. Rein D.W. (1977) Low-energy properties of a relativistic quark model with linear scalar potential. *Nouvo Cim*., **38 A**: 19.
3. Gestesy F. and Pittner L. (1978) Electrons in logarithmic potentials II: solution of the dirac equation. *J.Phys*., **11A**: 687.
4. Khelashvili A.A. (1981) Relativistic equations in case of infinitely rising potentials. *Bull. Georg. Natl. Acad. Sci*., **104**: 569.
5. Natroshvili K.R., Khelashvili A.A., Khmaladze V.Yu. (1985) On the infrared behavior of the gluon propagator in the Yang-Mills theory. *TMF*., **65**: 360.
6. Khelashvili A.A. (1982) Radial quasipotential equation for fermion and antifermion and infinitely rising central potentials. *TMF*., **51**: 201.
7. Krolikovski W. (1979) The absence of Klein paradox from Salpeter equation. *Acta Phys. Austriaca*., **51**: 243.
8. Turski A. (1979) The Klein paradox in the modified Dirac Equation. *Bull de I'acad. Polon. Des Science*., **27**: 195.
9. Logunov A.A. and Tavkhelidze A.N (1963) Quasipotential approach in quantum field theory. *Nouvo Cim*., **29**: 380.
10. Khelashvili A.A. (1969) Quasipotential equation for two particle systems with ½ spin. *Preprint of JINR*, **P2**: 4327. *Dubna*.
11. Faustov R.N. (1966) Solvable model of the relativistic two-fermion bound-state. *Nucl. Phys*., **75**: 669.
12. Breit G. (1930) The fine structure of He as a test of the spin interactions of two electrons. *Phys. Rev*., **36**: 483.
13. Kummer W. (1964) Exact solutions of the Bethe-Salpeter equation for fermions. *Nouvo Cim*., **31**: 219.
14. Silagadze Z.K. and Khelashvili A.A. (1984) Model of relativistic problem of two fermions bound state with confinement potentials. *TMF*., **61**: 431.
15. Sucher J. (1995) Confinement in relativistic potential models. *Phys. Rev*. **D51**: 5965.
16. Kevlishvili N., Khelashvili A. and Nadareishvili T. (2003) Once again about the Klein paradox. ArXiv: hep-th/**0312108.**

Received February, 2020