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Mathematics

An Extension of some Calderón-Zygmund Inequality

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The present paper deals with the study of behavior of the solution of Poisson equation with the free term from weighted grand Lorentz spaces. *© 2020 Bull. Georg. Natl. Acad. Sci.*

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In the present paper we consider the Poisson equation

$$\Delta u(x) = f(x), \quad \text{a. e. } x \in \Omega, \tag{1}$$

where Ω is a bounded Lipschitz domain in R^n, $n \geq 3$

When $f \in L^p(\Omega)$, $1 < p < \infty$, the existence and behavior of the solution of (1) was studied in [1] (see §9.4). The corresponding result is named as the Calderón-Zygmund inequality. An extension of the latter result to some new function spaces is done in [2], Section 14.9.

Let in recall the definition of weighted Lorentz spaces. Suppose that w is a weight function given on Ω i. e. w is a. e. positive and integrable on Ω.

Weighted Lorentz space is a set of measurable functions $f : \Omega \to R^1$ for which the norm

$$\|f\|_{L_w^{p,s}(\Omega)} = \left(s \int_0^\infty \left(w\{x \in \Omega : |f(x)| > \tau\} \right)^{\frac{s}{p}} \tau^{s-1} d\tau \right)^{1/s}, \quad 1 \leq p < \infty, \ 1 \leq s < \infty$$

is finite.

Here for the Borel e we set

$$we = \int_e w(x)\,dx.$$

It is easy to see that $L_w^{p,p}(\Omega)$ coincides with the weighted Lebesgue spaces.

On the base of the above mentioned function space we defined the weighted grand Lorentz spaces.

Let $1 < p < \infty$, $1 \leq s < \infty$ and let $\theta > 0$. For a measurable f and weight w let

$$\|f\|_{L_w^{p),s,\theta}(\Omega)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-1}} \|f\|_{L_w^{p-\varepsilon,s}(\Omega)}.$$

The spaces $L_w^{p),s,\theta}(\Omega)$ are non-reflexive, non-separable Banach function spaces. In [3] the extrapolation problem and application to the mapping properties of fundamental integral operators of Harmonic Analysis in weighted grand Lorentz spaces are explored. In the sequel the different constants independent of preimages f will be denoted by c.

In the sequel for every multi-index α with $|\alpha| \le k$, we use the mixed partial derivatives

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

With respect to the equation (1) the following statement holds.

Theorem 1. Let $1 < p, s < \frac{n}{2}$ and let $\theta > 0$. Then for $f \in L^{p),s,\theta}(\Omega)$ equations (1) is solvable and the solution $u \in L^{q),s',\frac{q}{p}\theta}(\Omega)$, where $\frac{1}{p} - \frac{1}{q} = \frac{2}{n}$ and $s' = \frac{s}{n-2s}$.

Furthemore,

i) $\|u\|_{L^{q),s',\frac{q}{p}\theta}(\Omega)} \le c\|f\|_{L^{p),s,\theta}(\Omega)}$

ii) $\|D^1 u\|_{L^{r),s',\frac{r\theta}{p}}(\Omega)} \le c\|f\|_{L^{p),s,\theta}(\Omega)}$ when $\frac{1}{p} - \frac{1}{r} = \frac{1}{n}$, $1 < s < n$ and $s' = \frac{s}{n-s}$.

iii) $\|D^2 u\|_{L^{p),s,\theta}(\Omega)} \le c\|f\|_{L^{p),s,\theta}(\Omega)}$.

Definition. A weight function $w : \Omega \to R^1$ is said to be of Muckenhoupt class $A_p(\Omega)$ if

$$\sup_B \left(\frac{1}{|\Omega \cap B|} \int_{\Omega \cap B} w(x)\,dx \right) \left(\frac{1}{|\Omega \cap B|} \int_{\Omega \cap B} w^{1-p'}(x)\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls with centers in Ω and $p' = p/(p-1)$.

Theorem 2. Let $1 < p, s < \frac{n}{2}$ and let $\theta > 0$. Suppose that $w \in A_{1+\frac{q}{p'}}$. Then for $f \in L_w^{p),s,\theta}$ the equation (1) is solvable and the solution u belongs to $f \in L_w^{q),s',\theta}(\Omega)$, where $\frac{1}{p} - \frac{1}{q} = \frac{2}{n}$ and $s' = \frac{s}{n-2s}$. Furthemore

$$\|u\|_{L_w^{q),s',\frac{q}{p}\theta}(\Omega)} \le c\|f\|_{L_w^{p),s,\theta}(\Omega)}.$$

Theorem 3. Let $1 < p, s < \frac{n}{2}$ and let $\theta > 0$. Suppose that $w \in A_p(G)$. Then for the derivatives of the solution of (1) we have the following estimates:

i) $\|D^1 u\|_{L_w^{q),s',\frac{r\theta}{p}}} \le c\|f\|_{L_w^{p),s,\theta}}$ where $\frac{1}{p} - \frac{1}{r} = \frac{1}{n}$, $1 < s < n$ and $s' = \frac{s}{n-s}$.

ii) $\left\|D^2 u\right\|_{L_w^{p),s,\theta}(\Omega)} \le c\left\|f\right\|_{L_w^{p),s,\theta}}$.

Here we need to notice that

$$A_p(\Omega) \subset A_{1+\frac{r}{p'}}(\Omega) \subset A_{1+\frac{q}{p'}}(\Omega).$$

Finally we mention that the proofs of afore said theorems are based on the integral representation of the solution u and boundedness results on singular and fractional integral operators established in [3].

მათემატიკა

კალდერონ-ზიგმუნდის ერთი თეორემის განზოგადების შესახებ

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