საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 14, №4, 2020
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 14, no. 4, 2020

Mathematics

The Consistent Estimators of Parametres of Gaussian Statistical Structures

Zurab Zerakidze[*], Lela Aleksidze[*], Laura Eliauri[*]

[*]*Faculty of Mathematics, Gori State Teaching University, Gori, Georgia*

(Presented by Academy Member Elizbar Nadaraya)

This paper provides necessary and sufficient conditions for the existence of consistent estimators of the parameters of Gaussian statistical structures. For this purpose two definitions of strongly separable structure are introduced. It is proven that necessary and sufficient condition for the existence of consistent estimators of the parameters of countable Gaussian statistical structures is their strong separation by the first definition. Besides, Borel orthogonal statistical structures are investigated and it is shown that in this case, necessary and sufficient condition for the existence of consistent estimators of the parameters is a strong separation of some statistical structure by the second definition. *© 2020 Bull. Georg. Natl. Acad. Sci.*

Consistent estimators, singular structure, weakly separable structure, strongly separable structure

Let (E,S) be a measurable space and there is given the family of probability measures $\{\mu_i, i \in I\}$ defined on S (I is the set of parameters).

Let us bring some definition (see [1] – [3]).

Definition 1. The set of objects $\{E, S, \mu_i, i \in I\}$ (where (E,S) is a measurable space) is called a statistical structure.

Definition 2. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal (singular) if μ_i and μ_j are orthogonal for each $i \neq j$, $i \in I$, $j \in I$.

Definition 3. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable if there exists a family of S -measurable sets $\{X_i, i \in I\}$ such that the relations are fulfilled:

$$\mu_i(X_j) = \begin{cases} 1, & if \quad i = j; \\ 0, & if \quad i \neq j. \end{cases}$$

Definition 4. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called separable if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that the relations are fulfilled:

$$\mu_i(X_j) = \begin{cases} 1, & if \quad i = j; \\ 0, & if \quad i \neq j. \end{cases}$$

$$card(X_i \cap X_j) < c, \quad if \quad i \neq j,$$

where c denotes the continuum power.

Definition 5. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly separable if there exists a disjoint family of S-measurable sets $\{X_i, i \in I\}$ such that the relations are fulfilled:

$$\mu_i(X_i) = 1, \quad \forall i \in I.$$

Remark 1. It is known that strong separability implies weak separability, and weak separability implies orthogonality, but not vice versa.

Let I be a set of parameters and let $B(I)$ be a σ-algebra of subsets of which contains all its finite subsets.

Definition 6. We say that the statistical structure $\{E, S, \mu_i, i \in I\}$ admits a consistent estimator of parameters $i \in I$, if there exists at least one measurable mapping

$$\delta : (E, S) \to (I, B(I)),$$

such that

$$\mu_i(\{x : \delta(x) = i\}) = 1, \quad \forall i \in I.$$

Let (E, S) be a measurable space, S_n is increasing sequence of σ-algebras such that $\bigcup_{n=1}^{\infty} S_n = S$. Let I be a matric space with metric ρ.

Definition 7. We say that the statistical structure $\{E, S, \mu_i, i \in I\}$ admits a weakly consistent estimator of parameters $i \in I$, if there exists sequence of S_n-measurable functions $g_n(x) : S_n \to I$ such that

$$\lim_{n \to \infty} \mu_i \{x : \rho(g_n(x), i) \geq \varepsilon\} = 0, \quad \forall i \in I.$$

Definition 8. We say that the statistical structure $\{E, S, \mu_i, i \in I\}$ admits a strongly consistent estimator of parameters $i \in I$, if there exists sequence of S_n-measurable functions $g_n(x) : S_n \to I$ such that

$$\mu_i \{x : \lim_{n \to \infty} \rho(g_n(x), i) = 0\} = 1, \quad \forall i \in I.$$

Remark 2. It is known that (see [1]): 1) if the statistical structure $\{E, S, \mu_i, i \in I\}$ admits a weakly consistent estimator of parameters $i \in I$, then this statistical structure is weakly separable; 2) if the statistical structure $\{E, S, \mu_i, i \in I\}$ admits a strongly consistent estimator of parameters $i \in I$, then this statistical structure admits a consistent estimator of parameters $i \in I$; 3) if the statistical structure $\{E, S, \mu_i, i \in I\}$ admits a consistent estimator of parameters $i \in I$, then this statistical structure is strongly separable, but not vice versa.

Example 1. As a set of parameters consider the set $I=(-\infty,+\infty)$ and let $B(H)=L(R)$ be Lebesgue σ-algebra on R. Let $\delta: R\to R$ denote some bijective mapping at the axis R which is Lebesgue non-measurable. We divide the segment $\left[-\frac{1}{2},\frac{1}{2}\right]$ into classes as follows: points x and y are included in a certain class if and only if the difference $x-y$ is a rational number. It is evident that the different classes are disjoint. Let us take one point from each class and mark the set of these points with A. It is obvious that the set A is not $L(R)$-measurable and its cardinality is continuum $cardA=c$. Hence, there exists one to one mapping $f_1: A\to[0,1]$ such that $f_1(A)=[0,1]$. As $A\subset\left[-\frac{1}{2},\frac{1}{2}\right]\subset[-1,1]$, it is obvious that $card\{[-1,1]\setminus A\}=c$ and there exists the bijective reflection $f_2:[-1,1]\setminus A\to[-1,0]$ such that $f_2([-1,1]\setminus A)=[-1,0]$. Let

$$\delta(x)=\begin{cases} x, & if \quad x\in R\setminus[-1,1]; \\ f_1, & if \quad x\in A; \\ f_2, & if \quad x\in[-1,1]\setminus A. \end{cases}$$

$\delta(x)$ is Lebesgue non-measurable because $\delta^{-1}[0,1]=f_1^{-1}[0,1]=A$. Hence, the inverse mapping δ^{-1} will also be Lebesgue non-measurable. Let

$$\mu_i(X)=\begin{cases} 1, & if \quad \delta(i)\in X; \\ 0, & if \quad \delta(i)\notin X, \end{cases}$$

for $i\in R$ and $X\in L(R)$. It is easy to see that the statistical $\{R,L(R),\mu_i,i\in R\}$ is strongly separable statistical structure that does not admit a consistent estimator of parameters.

The Consistent Estimators for Gaussian Statistical Structure

Let $\xi_i(t,\omega)$, $t=(t_1,t_2,...,t_n)\in T$, where T is a closed bounded subset of R^n, $i\in I$ be a real Gaussian homogeneous field on T with zero mean $E[\xi_i(t)]=0$ ($t\in T$, $i\in I$) and correlation functions of a difference of arguments $E[\xi_i(t)\xi_i(s)]=R_i(t-s)$ ($t,s\in T$, $i\in I$). Let $\{\mu_i, i\in I\}$ be the corresponding probability measures given on (E,S) and let $f_i(\lambda)$, $\lambda\in R^n$ be bounded densities for all $i\in I$. Let

$$\int_{R^n}\int_{R^n}\frac{|\tilde{b}_{ij}(\lambda,\mu)|^2}{f_i(\lambda)f_i(\mu)}d\lambda d\mu=\infty,\ \forall i\neq j,\ i,j\in I,$$

where $\tilde{b}_{ij}(\lambda,\mu)$ ($\lambda,\mu\in R^n$, $\forall i\neq j$, $i,j\in I$) is the Fourier transformation of the following functions $b_{ij}(t,s)=R_j(t,s)-R_i(t,s)$ ($\forall i\neq j$, $i,j\in I$).

Then the corresponding probability measures μ_i and μ_j are pairwise orthogonal measures (see [2]) and $\{E,S,\mu_i,i\in I\}$ is a Gaussian orthogonal statistical structure.

Theorem 1. The Gaussian statistical structure $\{E,S,\mu_i,i\in N\}$ ($N=\{1,2,...\}$) admits a consistent estimator of parameters if and only if

$$\int_{R^n}\int_{R^n}\frac{|\tilde{b}_{ij}(\lambda,\mu)|^2}{f_i(\lambda)f_i(\mu)}d\lambda d\mu=+\infty,\ \forall i\neq j,\ i,j\in I.$$

Proof. Necessity. Since the Gaussian statistical structure $\{E,S,\mu_i, i\in I\}$ $(cardI=\aleph_0)$ admits a consistent estimator of parameters, there exists a measurable mapping $\delta:(E,S)\to(I,B(I))$, such that

$$\mu_i(\{x:\delta(x)=i\})=1,\ \forall i\in I\ \ (cardI=\aleph_0).$$

Let $X_i=\{x:\delta(x)=i\}$, $\forall i\in I$. Therefore, the statistical structure $\{E,S,\mu_i, i\in I\}$ is strongly separable. From strong separability follows orthogonality and

$$\int_{R^n}\int_{R^n}\frac{|\tilde{b}_{ij}(\lambda,\mu)|^2}{f_i(\lambda)f_i(\mu)}d\lambda d\mu=+\infty,\ \forall i\neq j,\ i,j\in I.$$

Sufficiency. Let

$$\int_{R^n}\int_{R^n}\frac{|\tilde{b}_{ij}(\lambda,\mu)|^2}{f_i(\lambda)f_i(\mu)}d\lambda d\mu=+\infty,\ \forall i\neq j,\ i,j\in I.$$

The singularity of probability measures implies the existence of a family of S-measurable sets X_{ij} such that for any $i\neq j$ we have $\mu_j(X_{ij})=0$ and $\mu_i(E\setminus X_{ij})=0$. If we now consider the sets $X_i=\bigcup_{i\neq j}(E\setminus X_{ij})$ we will see that $\mu_i(X_i)=0$ and $\mu_j(E\setminus X_i)=0$, $\forall j\neq i$. It means that the S-measurable sets $\{X_i, i\in I\}$ such that

$$\mu_j(X_i)=\begin{cases}1, & if\quad i=j;\\ 0, & if\quad i\neq j.\end{cases}$$

Let us consider the sets $\overline{X}_i=X_i\setminus(X_i\cap(\bigcup_{j\neq i}(X_j)),\ i\in I$. It is clear that $\overline{X}_i\cap\overline{X}_j$, $\forall i\neq j$ and $\mu_i(\overline{X}_i)=1$, $\forall i\in I$.

Define the mapping $\delta:(E,S)\to(I,B(I))$ as follows: $\delta(\overline{X}_i)=1$, $i\in I$. Then we have $\mu_i(\{x:\delta(x)=i\})=1$, $\forall i\in I$, i.e. the statistical structure $\{E,S,\mu_i, i\in I\}$ $(cardI=\aleph_0)$ admits a consistent estimator of parameters.

Remark 3. From Theorem 1 it follows that for countable statistical structures $\{E,S,\mu_i, i\in N\}$ $(N=\{1,2,...\})$ the concepts of strong separability, separability, weak separability, orthogonality and the existence of a consistent estimator of parameters are equivalent.

The Consistent Estimators of Parameters

Let $\{\mu_i, i\in I\}$ be probability measures defined on the measurable space (E,S). For each $i\in I$ denote by $\overline{\mu}_i$ the completion of the measure μ_i, and denote by $dom(\overline{\mu}_i)$ the σ-algebra of all $\overline{\mu}_i$-measurable subsets of E. Let

$$S_1=\bigcap_{i\in I}dom(\overline{\mu}_i).$$

Definition 9. A statistical structure $\{E,S_1,\overline{\mu}_i, i\in I\}$ is called strongly separable if there exists the family of S_1-measurable sets $\{Z_i, i\in I\}$ such that the relations are fulfilled:

1) $\mu_i(Z_1)=1, \ \forall i \in I$

2) $Z_i \bigcap Z_j = \varnothing$, $\forall i \neq j$, $i,j \in I$;

3) $\bigcup_{i\in I} Z_i = E$.

Definition 10. We will say that the orthogonal statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ admits a consistent estimator of parameters if there exists at least one measurable mapping $\delta : (E, S_1) \to (I, B(I))$, such that

$$\overline{\mu}_i(\{x : \delta(x) = i\}) = 1, \quad \forall i \in I.$$

Let E be a complete separable metric space, $S_1 = \bigcap_{i\in I} dom(\overline{\mu}_i)$ a Borel σ-algebra of E, $cardI = c$.

Theorem 2. In order that the Borel orthogonal statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ ($cardI = c$) admitted a consistent estimator of parameters it is necessary and sufficient that this statistical structure was strongly separable.

Proof. Necessity. Since the Gaussian statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ ($cardI = c$) admits a consistent estimator of parameters, there exists a measurable mapping $\delta : (E, S_1) \to (I, B(I))$, such that

$$\overline{\mu}_i(\{x : \delta(x) = i\}) = 1, \quad \forall i \in I.$$

Let $X_i = \{x : \delta(x) = i\}$, $\forall i \in I$. Because $\overline{\mu}_i(X_i) = 1$ and $\bigcup_{i\in I} X_i = E$, therefore, the statistical structure is strongly separable.

Sufficiency. Since the statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$, $cardI = c$, is strongly separable, there exists a family $\{Z_i, i \in I\}$ of elements of the σ-algebra S_1 such that:

1) $\mu_i(Z_1)=1,, \ \forall i \in I$;

2) $Z_i \bigcap Z_j = \varnothing$, $\forall i \neq j$, $i,j \in I$;

3) $\bigcup_{i\in I} Z_i = E$.

For $x \in E$ we put $\delta(x) = i$, where i is from the set I for $i \in Z_i$.

Let now $Y \in B(I)$. Then $\{x : \delta(x) \in Y\} = \bigcup_{i\in I} Z_i$. We must show that $\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_i)$ for each $i \in I$.

If $i_0 \in Y$, then

$$\{x : \delta(x) \in Y\} = \bigcup_{i\in Y} Z_i = Z_{i_0} \bigcup \Big(\bigcup_{i\in Y\setminus\{i_0\}} Z_i \Big).$$

On the other hand, the validity of the condition

$$\bigcup_{i\in Y\setminus\{i_0\}} Z \subseteq \left(E \setminus Z_{i_0}\right)$$

implies that

$$\overline{\mu}_{i_0}\left(\bigcup_{i\in Y\setminus\{i_0\}} Z_i\right)=0\,.$$

The last equality yields that $\bigcup_{i\in Y\setminus\{i_0\}} Z_i \in dom(\overline{\mu}_{i_0})$.

Since $dom(\overline{\mu}_{i_0})$ is a σ -algebra, we deduce that

$$\{x:\delta(x)\in Y\}=Z_{i_0}\cup(\bigcup_{i\in Y\setminus\{i_0\}} Z_i)\in dom(\overline{\mu}_{i_0})\,.$$

If $i_0 \notin Y$, then

$$\{x:\delta(x)\in Y\}=\bigcup_{i\in Y} Z_i \subseteq (E\setminus Z_{i_0})$$

and we conclude that $\overline{\mu}_{i_0}(\{x:\delta(x)\in Y\})=0$. The last relation implies that

$$\{x:\delta(x)\in Y\}\in dom(\overline{\mu}_{i_0})\,.$$

Thus, we have shown the validity of the relation

$$\{x:\delta(x)\in Y\}\in dom(\overline{\mu}_{i_0})$$

for arbitrary $i_0 \in I$. Hence,

$$\{x:\delta(x)\in Y\}=\bigcap_{i\in I} dom(\overline{\mu}_i)=S_1\,.$$

All this means that the map $\delta:(E,S_1)\to(I,B(I))$ is measurable map. Since $B(I)$ contains all singletons of I we ascertain that

$$\overline{\mu}_i(\{x:\delta(x)=i\})=\overline{\mu}_i(Z_i)=1,\quad \forall i\in I\,.$$

მათემატიკა

პარამეტრების ძალდებული შეფასებები გაუსის სტატისტიკური სტრუქტურებისთვის

ზ. ზერაკიძე*, ლ. ალექსიძე*, ლ. ელიაური*

** მათემატიკის ფაკულტეტი, გორის სახელმწიფო სასწავლო უნივერსიტეტი, გორი, საქართველო*

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

სტატიაში შესწავლილია გაუსის სტატისტიკური სტრუქტურებისთვის პარამეტრების ძალდებული შეფასებების არსებობის აუცილებელი და საკმარისი პირობები. ამ მიზნით, მასში შემოტანილია ძლიერად განცალებადი სტატისტიკური სტრუქტურის ორი განმარტება. დამტკიცებულია, რომ თვლადი გაუსის სტატისტიკური სტრუქტურებისთვის პარამეტრების ძალდებული შეფასებების არსებობის აუცილებელი და საკმარისი პირობაა მათი ძლიერად განცალებადობა პირველი განმარტების აზრით. გარდა ამისა, განხილულია ბორელის ორთოგონალური სტატისტიკური სტრუქტურები და ნაჩვენებია, რომ ამ შემთხვევაში პარამეტრების ძალდებული შეფასებების არსებობის აუცილებელი და საკმარისი პირობაა გარკვეული სტატისტიკური სტრუქტურის ძლიერად განცალებადობა უკვე მეორე განმარტების აზრით.

REFERENCES

1. Ibramkhalilov I. and Skorokhod A. (1980) Consistent estimates of parameters of random processes, "*Naukova Dumka*", Kiev (in Russian).
2. Zerakidze Z. (1969) About the conditions equivalence of Gaussian measures corresponding to homogeneous fields, *Proceedings of Tbilisi State University*, 2: 215-220.
3. Zerakidze Z. (1986) Construction of statistical structures, *Theory of Probability & Its Applications*, XV, 3: 573-578.

Received July, 2020