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New Transient Solutions for some Semi-Markov Reliability Models

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Using a purely probabilistic reasoning, two classical semi-Markov models are investigated by means of the method of supplementary variables. This approach significantly simplifies the reliability analysis for considered systems. *© 2020 Bull. Georg. Natl. Acad. Sci.*

Probabilistic reasoning, semi-Markov process, failure, repair, Laplace transform

Semi-Markov models for systems reliability are considered to be essential in reliability engineering. In particular, semi-Markov models form an important theoretical part of Mathematical Theory of Reliability (MTR) [1-10].

At the same time, they have many practical applications in the industry and various technologies. Thus investigation of semi-Markov reliability models is the problem of primary importance. Specifically, in MTR the transient (time-dependent) solutions are of great theoretical value and very important for practical applications, as well.

Among classical theoretical approaches to investigation of semi-Markov processes the method of supplementary variables is considered to be the most productive [1-10].

Further development of that method in the direction of pure probabilistic argumentations, unlike traditional ones, is given in [5]. The object of consideration in [5] are semi-Markov queuing systems.

In this paper we illustrate how to apply this approach to some semi-Markov reliability models.

Classical Solutions for some Semi-Markov Reliability Models

In this section the classical solutions are given for two semi-Markov reliability models according to [2-4, 10]. We have only changed some notations for the convenience of our discussion.

One-unit system. We consider the one-unit unreliable, repairable technical system. It is repaired upon failure and then returned to operation. The unit is as good as a new one after repair. The unit's life-time is an exponential random variable with parameter α and repair time is a random variable σ with general distribution function G.

As known [2-4] the failure rate for the unit is α and for the repair rate μ has the form $\mu(x)=\frac{g(x)}{1-G(x)}$, where $g(x)=G'(x)$.

Let us introduce random processes $n(t)$ and $\xi(t)$, which define the states of the unit at time t. $n(t)$ is the number of failed units at time t. $n(t)=0$, if the unit is operative at time t and $n(t)=1$, if the unit failed (it is under repair); $\xi(t)$ is the repair time already received by the unit at time t.

It is clear that the process $n(t)$ is not Markovian, but the pair $(n(t),\xi(t))$ is two-dimensional Markov process. Thus, the analysis of a non-Markovian process is reduced to the analysis of two-dimensional Markov process. Such a method is referred to as the method of supplementary variables [1-5].

We denote by $P_0(t) = \mathbb{P}\{n(t) = 0\}$
and $p_1(x,t) = \lim_{h\to 0}\frac{1}{h}\mathbb{P}\{n(t) = 1, x \le \xi(x) < x + h\}$ (1)

Let us assume that this limit exists.

Using usual probabilistic reasonings we obtain the following system of equations [2,9]

$$P_o'(t)=-\alpha P_0(t)+\int_0^t p_1(x,t)\mu(x)dx, \tag{2}$$

$$\frac{\partial p_1(x,t)}{\partial t}+\frac{\partial p_1(x,t)}{\partial x}=-\mu(x)p_1(x,t). \tag{3}$$

There are the initial and boundary conditions

$$P_0(0)=1,\ p_1(x,0)=0,\ p_1(0,t)=\alpha P_0(t). \tag{4}$$

The equations (2) and (3) together with (4) constitute nonclassical boundary value problem of mathematical physics with non-local boundary conditions [4].

To solve the system (2-3) we introduce the function $q_1(x,t)$ in the following way

$$p_1(x,t)=q_1(x,t)(1-G(x)). \tag{5}$$

Then the equations (2) and (3) are reduced to the from

$$P_o'(t)=-\alpha P_0(t)+\int_0^t q_1(x,t)g(x)dx, \tag{6}$$

$$\frac{\partial q_1(x,t)}{\partial t}+\frac{\partial q_1(x,t)}{\partial x}=0 \tag{7}$$

and boundary condition is written as $q_1(0,t)=\alpha P_0(t)$.

Using Laplace transform to (7) and notation

$$\overline{q_1}(x,s)=\int_0^\infty e^{-st}q_1(x,t)dt$$

we obtain

$$s\overline{q_1}(x,s)+\frac{\partial \overline{q_1}(x,s)}{\partial x}=0. \tag{8}$$

It follows from (8) that

$$\overline{q_1}(x,s) = \overline{q_1}(x,0)e^{-sx}.$$

The factor e^{-sx} shows the delay and from boundary condition we have

$$q_1(x,t) = \begin{cases} aP_0(t-x), & t > x \\ 0, & t \le x \end{cases}. \tag{9}$$

Taking into consideration (8) the equation (2) may be written in the form

$$P_0'(t) = -\alpha P_0(t) + \alpha \int_0^t P_0(t-x)g(x)dx. \tag{10}$$

Using Laplace transform to (10) for initial condition $P_0(0) = 1$ we obtain

$$\overline{P_0}(s) = \left(s + \alpha\left(1 - \overline{g}(s)\right)\right)^{-1}, \tag{11}$$

where $\overline{P_0}(s) = \int_0^\infty e^{-st}P_0(t)dt$ and $\overline{g}(s) = \int_0^\infty e^{-st}g(t)dt$.

Finally, from (8) we have

$$\overline{p_1}(x,s) = \alpha\overline{P_0}(s)\left(1 - G(x)\right). \tag{12}$$

The Laplace transforms (11) and (12) give the solution of the above boundary value problem in terms of operational calculus. It is easy to obtain from them all numerical characteristics of the one-unit system.

Two-unit system. A redundant technical system consists of two identical, unreliable, repairable units. One of them is operative and another one is redundant. Life times for them are random variables with exponential distributions. Their failure rates are α_1 and α_2 respectively. The units' repair time is a random variable σ with general distribution function G. There is one repair server in the system.

When operative unit fails it is immediately replaced by redundant one, if it is not failed. Repair server immediately commences the repair of a unit, which fails first.

Let us start to analyze the reliability of the above system.

As we mentioned above repair rate (hazard function) has the form $\mu(x) = \frac{g(x)}{1-G(x)}$, $g(x) = G'(x)$.

We use here all our previous notations and add few new ones.

The process $n(t)$ now has three possible states: 0,1,2 (there are 0,1, or 2 failed units in the system).

The state $n(t) = 2$ is absorbing (failed) state Our aim is to analyze first passage time into the failed state as random variable η.

In addition of $P_0(t)$ and $p_1(x,t)$ we introduce a new notation.

$$P_2(t) = \mathbb{P}\{n(t) = 2\}.$$

It is clear, that $P_2(t)$ is at the same time the distribution function for random variable η.

The following equations are obtained according to [2,9].

$$P_0'(x,t) = -(\alpha_1 + \alpha_2)P_0(t) + \int_0^t e^{-\alpha_1 x} p_1(x,t)\mu(x)dx, \tag{13}$$

$$\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -\left[\alpha_1 + \mu(x)\right]p_1(x,t), \tag{14}$$

$$p_2'(t) = \alpha \int_0^t p_1(x,t)\,dx \,. \tag{15}$$

There are initial and boundary conditions

$$P_0(0) = 1, p_1(x,0) = 0, P_2(0) = 0, \tag{16}$$

$$p_1(0,t) = \alpha_2 P_0(t) \,. \tag{17}$$

We have again non classical boundary value problem with non-local boundary condition (13-17).

As a result of the solution of this problem in terms of operational calculus, the Laplace-Stieltjes transform for $P_2'(t)$ is obtained

$$L\{P_2'(t)\} = \frac{\alpha_1\alpha_2\left[1-\bar{g}(s+\alpha_1)\right]}{s+\alpha_2\left[1-\bar{g}(s+\alpha_1)\right](s+\alpha_2)} \,. \tag{18}$$

All numerical characteristics of the first passage time η can be easily obtained from (18).

New Probabilistic Solutions

In the models of stochastic systems a simple, purely probabilistic chance for their investigation is often "hidden" [4]. We offer here such a kind of novel approach to obtain the transient solutions to reliability models described above.

One-unit system. We prove here Theorem 1.

Theorem 1. The expression for the function $p_1(x,t)$ has the form:

$$p_1(x,t) = p_1(0,t-x)\left(1-G(x)\right). \tag{19}$$

Proof. Let us introduce events

$$A_1(x,t,h) = \{n(t) = 1; x < \xi(t) < x+h\} \,,$$

$$B_1(x) = \{\sigma > x\} \,,$$ (repair time is greater than *x*).

Using usual probabilistic argumentation, we have:

$$A_1(x,t,h) = A_1(0,t-x,h)\,B_1(x).$$

It is clear that from the definition of $p_1(x,t)$

$$p_1(x,t)h + o(h) = \mathbb{P}\{A_1(x,t,h).$$

Hence it follows from the independence of the events $A_1(x,t,h)$ and $B_1(x)$, that

$$p_1(x,t)h + o(h) = \mathbb{P}\{A_1(x,t,h) = \mathbb{P}\{A_1(0,t-x,h)[1-G(x)]. \tag{20}$$

Dividing both left and right parts of (20) by h and letting $h \to 0$, we obtain (1). As we see (19) gives a general form for a solution of the partial differential equation (2). The reader can easily verify this by the direct substitution of (19) into (2) instead of $p_1(x,t)$.

Two-unit system. We prove here the Theorem 2.

Theorem 2. The expression for the function $p_1(x,t)$ has the following form:

$$p_1(x,t) = p_1(0,t-x)e^{-\lambda x}\left[1-G(x)\right]. \tag{21}$$

As above we introduce the events $A_1(x,t,h) = \{n(t) = 1; x < \xi(t) < x+h\}$;

$B_1(x) = \{\sigma > x\}$ (repair time is greater than x) and

$C_1(x) = \{\text{active unit's life - time is greater than x}\}$.

It is easy to guess that $A_1(x,t,h) = A_1(0,t-x,h)B_1(x)\cdot C_1(x)$ and $P_1(x,t)h + o(h) = \mathbb{P}\{A_1(x,t,h)$.

Further from the independence of the events we have:

$$p_1(x,t)h = \mathbb{P}\{A_1(x,t,h)\} = \mathbb{P}\{A_1(0,t-x,h)B_1(x)\cdot C_1(x)\} =$$

$$= p_1(0,t-x,h)h\cdot\mathbb{P}\{B_1(x)\cdot\mathbb{P}\{C_1(x)\} = P_1(0,t-x,h)[1-G(x)]e^{-\alpha_1 x} + o(h)$$

From here we easily obtain (21).

We see again that (21) gives a general form for the solution of partial differential equation (14).

The Theorems 1 and 2 significantly simplify the final solutions to systems of equations (2-4) and (13-17) in terms of operational calculus. Apart from that, the expressions (19) and (21) are of great value for the solution of partial differential equations of type (3) (14) in general. It is known that such equations frequently arise in Mathematical Theory Reliability and other fields of research and technology. They are especially effective for a numerical solution of the boundary value problems formulated in the section 2.

But the main advantage of our approach is the fallowing fact: it allows us to obtain all reliability characteristics of considered of considered technical systems without using the partial differential (3) and (14), which are fundamental in classical models. To obtain these results, suffice to use equation (2) initial condition and boundary condition (4) for one-unit system and (13), (15) and (16) for two-unit system. The reader can easily verify these assertions.

Finally, as different from the requirement of equations (2) and (14) the function $p_1(x,t)$ to have partial derivatives with respect to t and x our approach does not require even the continuity of the $p_1(x,t)$.

Conclusion. The preceding discussion may be summed up by saying that in the stochastic models of reliability there is often "hidden" a simple, pure probabilistic chance for their investigation. Such a chance in this paper is formulated as Theorems 1 and 2. The basic point of the approach is consideration of the constructed semi-Markov processes simultaneously at two time-instants: 1) current time instant t and 2) the previous time instant $t-x$, where x is one of the possible values of a supplementary variable. Then using usual probabilistic reasoning the Theorems 1 and 2 are proved. They significantly simplify the reliability analysis of the considered technical systems [4]. Note, that the chosen systems are relatively simple ones, since our aim was to illustrate the possibility of the new approach only.

კიბერნეტიკა

საიმედოობის ზოგიერთი ნახევარმარკოვული მოდელის ახალი გადაწყვეტა გარდამავალ რეჟიმში

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