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Informatics

Fuzzy Relationships in an Urban System

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The paper presents algorithms for calculating the main characteristics of Q– analysis, obtained on the basis of the theory of fuzzy sets. An urban system is a collection of a finite number of objects: many different population groups (P), many people's interests (A), many buildings (B), many streets (S) etc. The structure of an urban system, in which various dynamic processes develop, is a set of mathematical relations $\wedge = \{\lambda, \mu ...\}$ that exist between these sets $A, B, S, P, ...$ Knowledge of the relations $\lambda, \mu ...$ allows us to discover the structural connection between them and to trace the development of the chain of connections between various functional objects in the space-time context. Usually, in order to clearly show whether a given relationship exists between any elements of these sets, we use the so-called Q-analysis method. In other words, to show the existence of a given relationship between the elements of a set, we use the characteristic function of any subset of the Cartesian multiplication of these sets. But, in many cases, if the connection between some elements of two sets is insignificant (small) in comparison with other connections, then when using the Q-analysis method, such a connection is considered zero. It is clear that further use of the model obtained with such an assumption loses a lot of information, which can lead to erroneous results, possibly even to a catastrophe. The approach presented by us allows us to overcome this problem by introducing fuzzy sets. In this case, the elements of the incidence matrix for Q-analysis are not 0 and 1, but the membership function of fuzzy sets. In the same way, to characterize the "connection" between elements of two sets, we use not the characteristic function of any subset of the Cartesian product of two sets, but the characteristic function of a fuzzy subset of the Cartesian product. *© 2021 Bull. Georg. Natl. Acad. Sci.*

Urban system, modeling, Q- analysis, fuzzy sets

An urban system is a collection of a finite number of objects: many different population groups (P), many interests of people (A), many buildings (B), many streets (S), etc.

The structure of an urban system, in which various dynamic processes develop, is a set of mathematical relations $\wedge = \{\lambda, \mu ...\}$ that exist between these sets $A, B, S, P, ...$

Knowledge of the relations $\lambda, \mu ...$ allows us to discover the structural connection between them and trace the development of a long chain of connections between objects of different functional purposes in a space-time context. For example, how actions taken in the service sector will affect infrastructure, transport capacity, demographics, environmental situation and much more. This allows us to systematically evaluate and predict all processes caused by our actions that are or may be launched in the city system.

Usually, in order to clearly show whether a given relationship exists between any elements of these sets, we use the corresponding incidence matrix of this relation – a matrix whose element is 1 (when there is a given relation between the corresponding elements) or 0 (when there is no given relation between the corresponding elements there), i.e. we use the so-called Q-analysis method [1]. In other words, to show the existence of a given relationship between the elements of two sets, we use the characteristic function of any subset of the Cartesian product of these sets.

But, in many cases, if the connection between some elements of two sets is insignificant (small) in comparison with other connections, then when using the Q-analysis method, such a connection is considered zero [2]. It is clear that further use of the model obtained with such an assumption loses a lot of information, which can lead to erroneous results, possibly even to a catastrophe.

The approach presented by us allows us to overcome this problem, in particular, to extend the Q-analysis by introducing fuzzy sets [3]. In this case, the elements of the incidence matrix are not numbers 0 and 1, but characteristic functions of fuzzy sets. In the same way, to characterize the “connection” between the elements of two sets, we will use not the characteristic function of any subset of the Cartesian product of two sets, but the characteristic function of a fuzzy subset of the Cartesian product. For example, if A is the number of city streets (or different buildings), and B is the number of public transport routes, then to determine how convenient a particular route is for residents of a particular street (house), it is better to use an inappropriate characteristic function, and the characteristic functions of fuzzy subsets (for example: “convenient”, “inconvenient”, etc.).

We think that in situations like the one given in the example, the models built on this approach will allow us to better assess the structure of the system [4] and its resistance to various disturbances [5].

Suppose that $A = \{a_1; a_2; \ldots; a_s\}$ and $B = \{b_1; b_2; \ldots; b_R\}$ – any sets, and we characterize the relationship between these sets by some linguistic variable. Let $X^1; X^2; \ldots; X^p$ denote the corresponding fuzzy subsets of the considered linguistic variable, and $\mu_p : A \times B \to [0;1]$ (for each $p \in \overline{[1;P]}$ – denotes the characteristic function of a fuzzy subset X^p. We assume that for each pair $(a_s; b_r)$ there exists at least one $p \in \overline{[1;P]}$ such that $\mu_p\ (a_s; b_r) = 1$. Let X_0^p denote the support of the characteristic function $\overline{\mu}_p$ the set of all pairs $(a;b) \in A \times B$ satisfying the condition $\mu_p(a;b) > 0$. Since our goal is to study the relationship between different subsets of the system under consideration, we consider the case when the condition is satisfied: $\bigcup_{p=1}^{p} X_0^p = A \times B$, however, it is possible to sort the sets X^p by levels. In other words, if $p_1 > p_2$ then X^{p_1} denotes a higher degree of connection than X^{p_2} (For example, if the relationship between the elements of the set A and B is characterized by the linguistic variable “connection”, fuzzy subsets can be: X^1 – “No connection”, X^2 – “Weak connection”, X^3 – “Medium strength connection”, X^4 – “Strong connection”).

We form the corresponding “fuzzy matrix” of the relation $\wedge$ as follows: for each $s \in \overline{[1;S]}$ $r \in \overline{[1;R]}$, the element at the intersection of the corresponding row and column of the matrix $\wedge$ (denoted by $\overrightarrow{(a_s; b_r)}$ symbol) is admittedly a vector:

$$\overrightarrow{(a_s; b_r)} = (\mu_1(a_s; b_r); \mu_2(a_s; b_r); \ldots; \mu_p(a_s; b_r)).$$

That is, the corresponding "fuzzy matrix" of the ratio λ between the elements of the set A and B:

$$\wedge=\begin{pmatrix}\overrightarrow{(a_1;b_1)} & \overrightarrow{(a_1;b_2)} & \dots & \overrightarrow{(a_1;b_R)}\\ \overrightarrow{(a_2;b_1)} & \overrightarrow{(a_2;b_2)} & \dots & \overrightarrow{(a_2;b_R)}\\ \dots & \dots & \dots & \dots\\ \overrightarrow{(a_S;b_1)} & \overrightarrow{(a_S;b_2)} & \dots & \overrightarrow{(a_S;b_R)}\end{pmatrix}.$$

Construct the vector of the corresponding structure of the fuzzy relation given by the "fuzzy matrix".

Let's compose the transposed matrix of this matrix:

$$\wedge^T=\begin{pmatrix}\overrightarrow{(a_1;b_1)} & \overrightarrow{(a_2;b_1)} & \dots & \overrightarrow{(a_S;b_1)}\\ \overrightarrow{(a_1;b_2)} & \overrightarrow{(a_2;b_2)} & \dots & \overrightarrow{(a_S;b_2)}\\ \dots & \dots & \dots & \dots\\ \overrightarrow{(a_1;b_R)} & \overrightarrow{(a_2;b_R)} & \dots & \overrightarrow{(a_S;b_R)}\end{pmatrix}.$$

We need to calculate the product of the matrices $\wedge$ and $\wedge^T$. To do this, we first need to determine the scalar product of two vectors by the set (we call the dot product of two vectors by the set, because we use it in accordance with the rules proposed by Zade [3]: if A and B are fuzzy sets and for each element x of the universal set $A(x), B(x), (A\cap B)(x)$ and $(A\cup B)(x)$ denote the value of the corresponding attribution function on x, then $(A\cap B)(x)=A(x)\wedge B(x)$, $(A\cup B)(x)=A(x)\vee B(x)$, where:

$$A(x)\wedge B(x)=\min\{A(x),B(x)\},\ A(x)\vee B(x)=\max\{A(x),B(x)\}):$$

$$\overrightarrow{(a_s;b_r)}\cdot\overrightarrow{(a_j;b_n)}=\max_{1\le p\le P}\{\min\{\mu_p(a_s;b_r);\mu_p(a_j;b_n)\}\},$$

In particular, when $n=r$, we have:

$$\overrightarrow{(a_s;b_r)}\cdot\overrightarrow{(s_j;b_r)}=\max_{1\le p\le P}\left\{\min\left\{\mu_p(a_s;b_r);\mu_p(a_j;b_r)\right\}\right\}.$$

Since we are considering sets with a finite number of elements, for each s, j and r there will be one or more $p\in\overrightarrow{1;P}$ for which the maximum is attained. Let $p_0(s,j,r)$ denote the largest of such p – s. Consider a fuzzy subset $X^{p_0(s,j,r)}$. The product of matrices $\wedge$ and $\wedge^T$ is called a fuzzy matrix, whose element c_{sj} is given by the formula:

$$c_{sj}=\max_{1\le r\le R}\left\{X^{p_0(s,j,r)}\right\}.$$

Since the subset X^p is sorted by levels, the element c_{ss} is a fuzzy set X^p, for which there is at least one element b_r in the set B such that $\mu_p(a_s;b_r)=^{\max}_{1\le k\le p}\{\mu_k(a_s;b_r)\}$, where p is the largest of these numbers. in other words, the element c_{ss} is the undefined subset of X^p that has the greatest "weight" for b_r, $(a_s;b_r)$ with p being the largest among such numbers;

If $s\ne j$, then c_{sj} is an undefined subset X^p for which there exists at least one element b_r from the set B such, that $\mu_p(a_s;b_r)=^{\max}_{1\le k\le p}\{\mu_k(a_s;b_r)\}$ and $\mu_p(a_j;b_r)=^{\max}_{1\le k\le p}\{\mu_k(a_j;b_r)\}$ where p is the largest such number.

Elements a_s and a_j are p-connectivity, if there is $m\ge p$ such that $c_{sj}=X^m$.

For each $p, 0\le p\le P$, construct the subset A^p of the set A in the following way: $a_s\in A^p$ if $c_{ss}=X^m$ and $m\ge p$. We represent the set A^p into subsets (equivalence classes) as follows: two elements a_s and a_j of the set A^p belong to the same equivalence class if and only, if they are p-connectivity or if there is some sequence of elements of the set A^p, the first member of this sequence is a_s, the last member is a_j, and any two subsequent members of this sequence are p-connectivity. Let Q_p denote the number of equivalence classes of the set A^p.

The vector $Q=(Q_0;Q_1;Q_2;...;Q_p)$ is called the vector of the structure of the given relation between the sets A and B.

Each Coordinate Q_p of the structure vector, for each $0\le p\le P$, shows `geometric obstacles` for the exchange of information on the level p (p of the degree of connection) between the elements of the set A^p. The larger the number Q_p, the greater the obstacle to establishing a level p connection between the elements of the set A^p.

Suppose that $p_0\in\overline{1;P}$. Let's introduce the notation:

$$\widehat{q}_s(p_0)=\#\{r: \text{ there is } p\ge p_0 \text{ such that } \mu_p(a_s;b_r)=1\}$$

(As noted above, for each pair $(a_s;b_r)$ there is at least one $p\in\overline{1;P}$ such that $\mu_p(a_s;b_r)=1$);

If $s\ne j$:

$$q_{sj}(p_0)=\begin{cases}\#\{\text{r : there exists } p_1\ge p_0, p_2\ge p_0 \text{ necessarily that } \mu_{p_1}(a_s;b_r)=1,\ \mu_{p_2}(a_j;b_r)=1.\\ 0, \quad \text{If such } r \text{ does not exist}\end{cases}$$

$$\breve{q}_s(p_0)=\max\{q_{sj}(p_0), j\in\overline{[1;S]}, j\ne s\}.$$

In other words, $\widehat{q}_s(p_0)$ shows how many different channels can be connected (exchange information) element a_s of set A by level p_0 or higher, with any other member of set A, whereas $\breve{q}_s(p_0)$ indicates the maximum number of channels at level p_0 or higher through which it actually connects to any other member of set A. $\widehat{q}_s(p_0)>\breve{q}_s(p_0)$ means that a_i is in some sense different from other elements.

The parameter calculated by the formula:

$$E_{CCp_0}(a_s)=\frac{\widehat{q}_s(p_0)-\breve{q}_s(p_0)}{\breve{q}_s(p_0)}$$

is called the eccentricity of the level P_0 of the element a_s of the set A.

The eccentricity of the p_0 level of the a_s element indicates the degree of "separation", the "uniqueness" of the a_s element. It is clear that the "uniqueness" of the a_s element, the degree of "separation" will be greatest when $\breve{q}_s(p_0)=0$ when the a_s element cannot communicate with another element (exchange information) even at the P_0 level.

ინფორმატიკა

არამკაფიო მიმართებანი ურბანულ სისტემაში

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(წარმოდგენილია აკადემიის წევრის რ. ხუროძის მიერ)

ურბანული სისტემა წარმოადგენს სასრული რაოდენობის ობიექტების ერთობლიობას: მაცხოვრებელთა სხვადასხვა ჯგუფთა სიმრავლე (P), ადამიანთა ინტერესების სიმრავლე (A), შენობათა სიმრავლე (B), ქუჩების სიმრავლე (S) და სხვა. ურბანული სისტემის სტრუქტურა, რომელზედაც ვითარდება სხვადასხვა დინამიკური პროცესები, წარმოადგენს მათემატიკურ მიმართებათა $\wedge = \{\lambda, \mu ...\}$ სიმრავლეს, რომლებიც არსებობენ ამ $A, B, S, P, ...$ სიმრავლეთა შორის. $\lambda, \mu ...$ მიმართებების ცოდნა საშუალებას გვაძლევს აღმოვაჩინოთ სტრუქტურული ბმულობა მათ შორის და თვალი ვადევნოთ სხვადასხვა ფუნქციური დანიშნულების ობიექტებს შორის ბმულობის ჯაჭვში განვითარებულ მოვლენებს, სივრცე-დროით ჭრილში.

ჩვეულებრივ, იმის თვალსაჩინოდ ჩვენებისათვის, არსებობს თუ არა ამ სიმრავლეების რაიმე ელემენტებს შორის მოცემული მიმართება, ვსარგებლობთ ე.წ. Q-ანალიზის მეთოდით. სხვა სიტყვებით რომ ვთქვათ, ორი სიმრავლის ელემენტებს შორის მოცემული მიმართების არსებობის თვალსაჩინოდ ჩვენებისათვის ვსარგებლობთ ამ სიმრავლეების დეკარტული ნამრავლის რაიმე ქვესიმრავლის მახასიათებელი ფუნქციით. მაგრამ, ხშირ შემთხვევაში, თუ ორ სიმრავლეთა გარკვეულ ელემენტებს შორის კავშირი არის უმნიშვნელო (მცირე) სხვა კავშირებთან შედარებით, მაშინ, Q-ანალიზის მეთოდის გამოყენებისათვის ასეთ კავშირს მიიჩნევენ ნულად. ცხადია, ასეთი დაშვებით მიღებული მოდელის შემდგომი გამოყენების დროს იკარგება უამრავი ინფორმაცია, რამაც შეიძლება მიგვიყვანოს არასწორ შედეგებამდე, შესაძლებელია კატასტროფამდეც. ჩვენ მიერ წარმოდგენილი მიდგომა, საშუალებას გვაძლევს დავძლიოთ აღნიშნული პრობლემა, არამკაფიო მიმართებების შემოტანით. ამ შემთხვევაში, Q-ანალიზისათვის ინცინდენტურობის მატრიცის ელემენტებია არა **0** და **1**, არამედ არამკაფიო სიმრავლეთა მიკუთვნების ფუნქციები. ეს იგივეა, რომ ორი სიმრავლის ელემენტებს შორის მიმართების, „კავშირის“ დასახასიათებლად ვსარგებლობთ არა ორი სიმრავლის დეკარტული ნამრავლის რაიმე ქვესიმრავლის მახასიათებელი ფუნქციით, არამედ დეკარტული ნამრავლის არამკაფიო ქვესიმრავლის მიკუთვნების ფუნქციით. ნაშრომში მოყვანილია Q-ანალიზის მნიშვნელოვანი მახასიათებლების გათვლის ალგორითმები, მიღებული არამკაფიო სიმრავლეთა თეორიისა და მიმართებათა საფუძველზე.

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