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Mathematics

Boundedness Criteria for Calderón Singular Integral Operator in Some Grand Function Spaces

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The paper deals with the study of mapping properties of Calderón singular integrals in generalized weighted grand Lebesgue spaces defined on a finite interval and on simple rectifiable curves. We present also the boundedness theorems in weighted grand Lebesgue spaces with mixed norms and grand variable exponent Lebesgue spaces without assuming the Log-Hòlder continuity conditions.

Calderón singular integral, generalized weighted grand Lebesgue space, Carleson curve, A_p weights

The grand Lebesgue space $L^{p)}$ was introduced in 1990 by Iwaniec and Sbordone [1] in the context of finding minimal conditions for the integrability of the Jacobian. This theory is nowadays one of the intensively developing directions in modern analysis. The necessity of introducing and studying these spaces grew of their rather essential role in various fields. We recall applications in partial differential equations (PDE), geometric function theory, Sobolev spaces theory and Banach function spaces theory.

In order to study non-homogeneous n-harmonic equation $div\, A(x,\nabla u)=\mu$, in [2] was introduced somewhat more general grand Lebesgue $L^{p),\theta}$ spaces. It turned out that in the theory of PDEs the generalized grand Lebesgue spaces are appropriate to the solutions of existence and uniqueness, and, also the regular problems for various nonlinear differential equations.

The boundedness problems for fundamental integral operators of Harmonic Analysis in grand Lebesgue spaces were intensively studied in the papers: [3-9], see, also, the monograph [10], Chapter 14 and references therein. In [9] we established the boundedness criteria for Calderón singular integral operator in weighted grand Lebesgue when the weight generates the absolute continuous measure in the definition of the norm.

In present paper we study the boundedness of Calderón singular integral operator in various grand function spaces. In generalized weighted grand Lebesgue spaces we investigate all cases of position of weights in definition of the norms: a weight function plays role of multiplier or create an absolute continuous measure of integration. For Calderón singular integral defined on rectifiable curve the boundedness criteria concern simultaneously the weight and curve. Finally we give the boundedness result for Calderón singular integral operator in grand variable exponent Lebesgue space introduced for the first time in [11].

Boundedness of Calderón Singular Integral Operator in Generalized Weighted Grand Lebesgue Spaces

Let (X,d,μ) be a quasi-metric space with a σ-finite complete measure μ and quasi-metric d. In what follows, we assume that $\mu X < +\infty$. Let w be a weight function i. e. positive, almost everywhere in the sense of measure μ and locally integrable on X. Let φ be a positive increasing and bounded on the interval $(0,p-1]$ and $\varphi(0+)=0$. Assume that $1<p<\infty$.

The generalized weighted grand Lebesgue space $L_w^{p),\varphi}$ is defined as the set of μ-measurable functions $f: X \to R^1$ with a finite norm

$$\|f\|_{L_w^{p),\varphi}} = \sup_{0<\varepsilon<p-1} \left(\varphi(\varepsilon) \int_X |f(x)|^{p-\varepsilon} w(x)\,dx \right)^{\frac{1}{p-\varepsilon}} \tag{1}$$

Along the space $L_w^{p),\varphi}$ we are interested in the space which is defined by a norm, where w plays a role of multiplier. The space $\mathcal{L}_w^{p),\varphi}$ is defined by the norm

$$\|f\|_{\mathcal{L}_w^{p),\varphi}} = \sup_{0<\varepsilon<p-1} \left(\varphi(\varepsilon) \frac{1}{\mu X} \int_X |f(x) w(x)|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}} < +\infty \tag{2}$$

Both of these spaces are non-reflexive, non-separable and non-rearrangement when $w(x) \not\equiv 1$ the Banach function spaces. Differing from classical Lebesgue spaces $L_w^{p),\varphi}$ and $\mathcal{L}_w^{p),\varphi}$ are not reducible each to another.

By definition a weight function w defined on X belongs to the class $A_p(X)$, $1<p<\infty$ if

$$\sup_{0<\varepsilon<1} \left(\frac{1}{\mu B} \int_B w(x)\,d\mu \right) \left(\frac{1}{\mu B} \int_B w^{1-p'}(x)\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls in X.

Let $X = I$ be the finite interval on R^1, μ be Lebesgue measure and d-Euclidean distance. Our goal is to study the boundedness of Calderón singular integral operator

$$C_I f(x) = \int_I \frac{a(x)-a(y)}{(x-y)^2} f(y)\,dy$$

in both spaces $L_w^{p),\varphi}$ and $\mathcal{L}_w^{p),\varphi}$.

The following statements are true:

Theorem 1. Let $1<p<\infty$ and let $a \in Lip\ 1$. Then we have:

Let $w \in A_p(I)$ then C_I is bounded in $L_w^{p),\varphi}(I)$;

If there exists a number m such that $0 < m \le |a'(t)|$ and C_I is bounded in $L_w^{p),\varphi}(I)$ then $w \in A_p(I)$.

Theorem 2. Let $1 < p < \infty$ and let $a \in Lip\ 1$ on I. Then we have:

The condition $w^p \in A_p$ ensures the boundedness of C_I in $\mathcal{L}_w^{p),\varphi}(I)$;

If there exists a number m, such that $0 < m \le |a'(x)|$ and C_I is bounded in $\mathcal{L}_w^{p),\varphi}(I)$, then $w^p \in A_p(I)$.

Calderón Singular Integral Defined on Rectifiable Curves and its Boundedness

Let $\Gamma = \{\mathbb{C}:\ t = t(s),\ 0 \le s < l < +\infty\}$ be a simple rectifiable curve with finite arc-length measure. In the remainder of this section, we use the notation

$$D(t,r) = \{z \in \mathbb{C}:\ |z - t| < r\}.$$

A rectifiable curve Γ is called a Carleson (regular) curve if

$$\sup_{\substack{0<r<diam\ \Gamma \\ t\in\Gamma}} \frac{|D(t,r)|}{r} < \infty.$$

Here $|D(t,r)|$ denotes the arc-length measure of portion $D(t,r)$.

Let now $X = \Gamma$, μ be an arc-length measure on Γ, d be the Euclidean distance. The spaces $L_w^{p),\varphi}(\Gamma)$ and $\mathcal{L}_w^{p),\varphi}(\Gamma)$ are defined by (1) and (2) respectively.

In the following we also need the definition of Muckenhoupt type weights suited to the rectifiable curves. We set:

$$A_p(\Gamma) = \left\{ w:\ \sup_{\substack{0<r<diam\ \Gamma \\ t\in\Gamma}} \left(\frac{1}{r} \int_{D(t,r)} w(\tau)\,ds \right) \left(\frac{1}{r} \int_{D(t,r)} w^{1-p'}(\tau)\,ds \right)^{p-1} < +\infty \right\}.$$

Let us consider the operator:

$$C_\Gamma(f)(t) = \int_\Gamma \frac{a(t) - a(\tau)}{(t-\tau)^2} f(\tau)\,d\tau.$$

We are able to prove the following statements:

Theorem 3. Let $1 < p < \infty$ and let $a \in Lip_1(\Gamma)$. Then we have:

The condition $w \in A_p(\Gamma)$ guarantees boundedness of C_Γ in $L_w^{p),\varphi}$ if Γ is a Carleson curve;

If there exists a number m such that $0 < m \le |a'(t)|$, then from the boundedness of C_Γ defined on some rectifiable curve in $L_w^{p),\varphi}$ follows that Γ is a Carleson curve and $w \in A_p(\Gamma)$.

Theorem 4. Let $1 < p < \infty$ and let $a \in Lip_1(\Gamma)$. Then we have:

The condition $w^p \in A_p(\Gamma)$ follows that C_Γ is bounded in $\mathcal{L}_w^{p),\varphi}$;

Suppose that there exists a number m such that $0 < m \le |a'(t)|$, then from the boundedness of C_Γ in $\mathcal{L}_w^{p),\varphi}(\Gamma)$ follows that Γ is a Carleson curve and $w^p \in A_p(\Gamma)$.

Boundedness of Multiple Singular Integral Operator's in Generalized Weighted Grand Lebesgue Spaces with Mixed Norms

Let $J = I_1 \times I_2$, where I_i $(i = 1,2)$ be the finite intervals. Assume that $1 < p_i < \infty$, w_i $(i = 1,2)$ are the weights on I_i. Then suppose that the functions φ_j $(j = 1,2)$ satisfy the similar conditions as the function φ in definition of $L^{p),\varphi}$. Let $P = (p_1, p_2)$, $\Phi = (\varphi_1, \varphi_2)$ and $W = (w_1, w_2)$. The generalized weighted grand Lebesgue space $L_w^{P),\Phi}$ with mixed norm is the set of functions $f : J \to R^1$ for which the norm

$$\|f\|_{L_w^{P),\Phi}} = \sup_{\substack{0<\varepsilon_j<p_i-1 \\ j=1,2}} \prod_{i=1}^{2} \left(\varphi_i(\varepsilon_i)\right)^{\frac{1}{p_i-\varepsilon_i}} \left\| \|f(x,y)\|_{L_{w_2}^{p_2-\varepsilon_2}} \right\|_{L_{w_1}^{p_1-\varepsilon_1}}$$

is finite.

The notion of these spaces was introduced and studied in view of boundedness of some multiple integral transforms by the author in [12]. Now we will treat the boundedness problem for multiple singular integral operator

$$C_J f(x_1, x_2) = \int_J f(y_1, y_2) \prod_{j=1}^{2} \frac{a_j(x_j) - a_j(y_j)}{(x_j - y_j)^2} dy_1 dy_2$$

Theorem 5. Let $1 < p_j < \infty$, $a_j \in Lip_1(I_J)$. Then the following statements are true:

The conditions $w_j \in A_{p_j}(I_j)$ ensure the boundedness of C_J in $L_W^{P),\Phi}$;

If there exists a number m such that $0 < m \le |a_j(x)|$, $j = 1,2$, then from the boundedness of C_J follows that $w_j \in A_{p_j}(I_j)$.

Boundedness of Calderón Singular Integral Operator in Grand Variable Exponent Lebesgue Space without Assuming the Log-Hòlder Condition on the Exponent

For arbitrary interval on R^1 denote by $P(I)$ the family of all measurable functions p on I such that

$$1 < p_- \le p_+ < \infty,$$

where $p_- := p_-(I) := \inf_I p(x)$; $p_+ := p_+(I) := \sup_I p(x)$.

Let $p(\cdot) \in P(I)$. The variable exponent Lebesgue space $L^{p(\cdot)}(I)$ is the set of all measurable functions f on I for which the modular

$$S_{p(\cdot)}(f) = \int_T |f(x)|^{p(x)} dx$$

is finite.

The norm in $L^{p(\cdot)}(I)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(I)} = \inf\left\{\lambda > 0 : S_{p(\cdot)}\left(f/\lambda\right) \le 1\right\}.$$

It is known that $L^{p(\cdot)}(I)$ is a Banach function space.

Definition. Let $p(\cdot)$ be an exponent on R^1 such that $p(\cdot)\in P(R^1)$. They say that $p(\cdot)$ satisfies the condition $dx \in A_{p(\cdot)}$ if

$$\sup_I |I|^{-1} \|\chi_I\|_{L^{p(\cdot)}} \|\chi_I\|_{L^{p'(\cdot)}} < \infty ,$$

where $p'(\cdot)=\dfrac{p(\cdot)}{p(\cdot)-1}$ and the supremum is taken over all finite I in R^1.

Further by $M(I)$ we denote the class of exponents $p\in P(I)$ such that the Hardy-Littlewood maximal operator is bounded in $L^{p(\cdot)}(I)$.

Proposition [13]**.** Let $p(\cdot)$ be an exponent in R^1. Then $p(\cdot)\in M(R^1)$ if and only if $dx\in A_{p(\cdot)}$ provided that $p(\cdot)$ is constant outside some large interval in R^1.

Let now I be an finite interval. By definition the grand variable exponent Lebesgue space $L^{p(\cdot),\theta}(I)$ is the set of all measurable functions $f: I \to R^1$ for which the norm

$$\|f\|_{L^{p(\cdot),\theta}(I)} = \sup_{1<q<p_-} \left(\frac{p_-}{q'}\right)^{\frac{\theta q}{p_-}} \|f\|_{L^{p(\cdot)/q}(I)}$$

is finite.

The following statement is true:

Theorem 6. Let I be a finite interval in R^1. Suppose that $p(\cdot)$ is an exponent function on I such that there is its extension $P(\cdot)$ such that it is constant outside some large ball, and $dx\in A_{p(\cdot)}$. Then the operator C_I is bounded in $L^{p(\cdot),\theta}(I)$.

მათემატიკა

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