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Mathematics

Reduction of a Four-Layer Semi-Discrete Scheme for an Abstract Evolution Equation to Two-Layer Schemes and Estimation of the Approximate Solution Errors by Using Associated Polynomials

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In the paper, a purely implicit four-layer semi-discrete scheme for an abstract evolution equation is reduced by means of an perturbation algorithm to two-layers schemes. Based on the latter schemes, an approximate solution of the initial problem is constructed. The approximate solution error estimate is obtained in the Hilbert space by using associated polynomials. *© 2021 Bull. Georg. Natl. Acad. Sci.*

Evolution problem, semi-discrete scheme, perturbation algorithm

As we know a great deal of interest is shown in questions related to the construction and investigation of algorithms of approximate solution of evolution problems, including the problem of realization and investigation of many-layer schemes for the solution of these problems. The main difficulty that arises when realizing many-layer schemes (especially for multi-dimensional problems) consists in the necessity of using a large operational memory that increases proportionally to the growth of the number of layers. In our opinion, one of the possible ways to overcome this difficulty is the application of a perturbation algorithm for splitting the many-layer schemes. A perturbation algorithm applicable to difference schemes for differential equations was considered in the work [1]. The algorithm proposed there is very close to the methods considered in the works of G. I. Marchuk and V. V. Shaidurov [2] and V. Pereyra [3, 4].

In [5, 6], the realization of a purely implicit three-layer scheme for an evolution equation by using a perturbation algorithm is reduced to the realization of two-layer schemes. In these works, by means of semi-groups an implicit estimate is proved for the error of an approximate solution of the initial problem in the Banach space under sufficiently general assumptions on the problem data.

In the present paper, for an approximate solution of the Cauchy problem for an evolution equation with self-adjoint positive-definite operator we consider, in the Hilbert space, a purely implicit four-layer semi-discrete scheme which is reduced to two-layers schemes. Using the solutions of these schemes, an approximate solution of the initial problem is constructed. It should be noted that the first two-layer scheme gives an approximate solution to an accuracy of first order, while the solution of each next scheme defines the preceding solution more exactly by an order. To estimate the error of an approximate solution we use the approach proposed in [7] where the investigation of the stability of linear many-step methods is based on the properties of a class of polynomials in many variables (these are the so-called associated polynomials), which are a natural generalization of classical Chebyshev polynomials of the second kind.

Reduction of the Purely Implicit Four-Layer Scheme to Two-Layer Schemes

In the Hilbert space H we consider the following evolution problem

$$\frac{du(t)}{dt}+Au\left(t\right)=f(t),\quad t\in\left]0,T\right], \tag{1}$$

$$u\left(0\right)=u_0, \tag{2}$$

where A is a self-adjoint positive-definite in H with definition domain $D(A)$ everywhere dense in H; $f(t)$ is a continuously differential function taking values from H; u_0 is a given vector from H; $u(t)$ is the function to be defined.

Let us introduce on [0,T] the grid $t_k=k\tau$, $k=0,1,\ldots,n$, with step $\tau=T/n$. For the approximation of the first derivative we use a purely implicit four-layer scheme. As a result, equation (1) at the point $t=t_k$ can be rewritten in the following form

$$\begin{gathered}\frac{u(t_k)-u(t_{k-1})}{\tau}+Au\left(t_k\right)+\frac{\tau}{2}\frac{\Delta^2u(t_{k-2})}{\tau^2}+\frac{\tau^2}{3}\frac{\Delta^3u(t_{k-3})}{\tau^3}\\=f(t_k)-\tau^3R_k(\tau,u),\end{gathered} \tag{3}$$

where $k=3,\ldots,n,\ \Delta u(t_{k-1})=u(t_k)-u(t_{k-1}),\ R_k(\tau,u)\in H.$

From **Error! Reference source not found.**, using the perturbation algorithm [1], we obtain the following system of equations

$$\frac{u_k^{(0)}-u_{k-1}^{(0)}}{\tau}+Au_k^{(0)}=f_k,\qquad f_k=f(t_k),\ \ u_0^{(0)}=u_0,\ \ k=1,\ldots,n, \tag{4}$$

$$\frac{u_k^{(1)}-u_{k-1}^{(1)}}{\tau}+Au_k^{(1)}=-\frac{1}{2}\frac{\Delta^2u_{k-2}^{(0)}}{\tau^2},\qquad k=2,\ldots,n, \tag{5}$$

$$\frac{u_k^{(2)}-u_{k-1}^{(2)}}{\tau}+Au_k^{(2)}=-\frac{1}{2}\frac{\Delta^2u_{k-2}^{(1)}}{\tau^2}-\frac{1}{3}\frac{\Delta^3u_{k-3}^{(0)}}{\tau^3},\qquad k=3,\ldots,n, \tag{6}$$

Let us introduce the notation $v_k=u_k^{(0)}+\tau u_k^{(1)}+\tau^2u_k^{(2)},\qquad k=3,\ldots,n.$ (7)

Assume that the vector v_k is an approximate value of the exact solution of problem **Error! Reference source not found.,Error! Reference source not found.** for $t=t_k$, $u(t_k)\approx v_k$.

Note that in scheme **Error! Reference source not found.** we define the starting vector $u_1^{(1)}$ from the equality $v_1 = u_1^{(0)} + \tau u_1^{(1)}$, where $u_1^{(0)}$ is found by scheme **Error! Reference source not found.**, and v_1 is an approximate value of $u(t_1)$ to an accuracy of $O(\tau^3)$. Analogously, we define the starting vector $u_2^{(2)}$ from the equality $v_2 = u_2^{(0)} + \tau u_2^{(1)} + \tau^2 u_2^{(2)}$, where $u_2^{(0)}$ and $u_2^{(1)}$ are found by scheme **Error! Reference source not found.** and **Error! Reference source not found.** respectively, while a v_2 is an approximate value of $u(t_2)$ to an accuracy of $O(\tau^3)$.

We can show that the construction of v_k satisfies the following equation:

$$\frac{\frac{11}{6}v_k - 3v_{k-1} + \frac{3}{2}v_{k-2} - \frac{1}{3}v_{k-3}}{\tau} + Av_k = f_k + \tilde{R}_k(\tau), \quad k = 5,\dots,n, \tag{8}$$

where for the residual $\tilde{R}_k(\tau)$, the estimate

$$\left\|\tilde{R}_k(\tau)\right\| \le c\tau^3, \qquad c = const > 0, \qquad k = 5,\dots,n. \tag{9}$$

is valid for sufficiently smooth initial data.

The investigation of scheme **Error! Reference source not found.-Error! Reference source not found.** rests on some facts related to polynomials associated with a difference equation of higher order.

Polynomials Associated with a Difference Equation of Higher Order

The results presented in this subsection are, in our opinion, of independent interest. They have found an application in the investigation of some many-layer schemes [7].

Let us consider a difference equation of order $q \ge 1$ written in the following form

$$z_k - x_1 z_{k-1} - \dots - x_q z_{k-q} = 0, \quad k \ge q, \tag{10}$$

where x_i $(i = 1,\dots,q)$ are real numbers.

Lemma 1. For the solution of equation **Error! Reference source not found.** the formula

$$z_{k+q-1} = U_k z_{q-1} + \left(x_2 U_{k-1} + \dots + x_q U_{k-q+1}\right) z_{q-2}$$
$$+ \left(x_3 U_{k-1} + \dots + x_q U_{k-q+2}\right) z_{q-3} + \dots + x_q U_{k-1} z_0, \quad k = 1,2,\dots,$$

is valid, where the polynomials $U_k(x_1,\dots,x_q)$ satisfy the recurrent relation

$$U_k(x_1,\dots,x_q) = x_1 U_{k-1}(x_1,\dots,x_q) + x_2 U_{k-2}(x_1,\dots,x_q) + \dots$$
$$+ x_q U_{k-q}(x_1,\dots,x_q), \quad k = 1,2,\dots, \tag{11}$$
$$U_0 = 1, \; U_{-1} = \dots = U_{1-q} = 0.$$

The lemma is proved by the method of mathematical induction.

We call the polynomials $U_k(x_1,\dots,x_q)$ the polynomials associated with the difference equation **Error! Reference source not found.**.

It is easy to observe that $U_k(2x,-1)$ $(q=2)$ are Chebyshev polynomials of the second kind. Hence it is natural to call the polynomials defined by the recurrent relation (11) Chebyshev polynomials in q variables.

Lemma 2. The formula

$$\lambda_i^{n+q-1} = \lambda_i^{q-1}U_n + \lambda_i^{q-2}\left(x_2U_{n-1} + \ldots + x_qU_{n-q+1}\right) + \ldots$$
$$+\lambda_i\left(x_{q-1}U_{n-1} + x_qU_{n-2}\right) + x_qU_{n-1}, \quad n = 1,2,\ldots \ , \tag{12}$$

is valid, where $\lambda_i (i=1,\ldots,q)$ are the roots of the characteristic equation

$$\lambda^q - x_1\lambda^{q-1} - \ldots - x_{q-1}\lambda - x_q = 0. \tag{13}$$

The proof rests on the fact that the difference equation (for the sake of simplicity we consider the case q=3)

$$z_{k+2} - x_1z_{k+1} - x_2z_k - x_3z_{k-1} = 0,$$

is equivalent to the system of equations

$$z_k - \lambda_1 z_{k-1} = z_k^1,$$
$$z_{k+1}^1 - \lambda_2 z_k^1 = z_{k+1}^2, \tag{14}$$
$$z_{k+2}^2 - \lambda_3 z_{k+1}^2 = 0,$$

where λ_i $(i=1,2,3)$ are the roots of equation (13).

For $z_1 = \lambda_1 z_0$, $z_2 = \lambda_1^2 z_0$ the solution of system **Error! Reference source not found.** is given by the formula

$$z_k = \lambda_1^k \ z_0 \quad (k=1,2,\ldots). \tag{15}$$

On the other hand, by virtue of Lemma 1 we have

$$z_{k+2} = U_kz_2 + \left(x_2U_{k-1} + x_3U_{k-2}\right)z_1 + x_3U_{k-1}z_0.$$

Hence, for $z_1 = \lambda_1 z_0, z_2 = \lambda_1^2 z_0$ and from **Error! Reference source not found.**, formula **Error! Reference source not found.** follows for $q=3$.

It is obvious that we can take as λ_1 any root of equation **Error! Reference source not found.**. Therefore formula **Error! Reference source not found.** holds for any i $(i=1,2,3)$.

The following statement is true.

Theorem 3. Let M be some set of points $P \in E^q$ ($P = (x_1,\ldots,x_q)$), E^q is a Euclidean space of dimension q). Assume that the following condition is fulfilled: for any $P \in M$ all the roots of equation **Error! Reference source not found.**, except perhaps one root, belong to one and the same circle lying within the unit circle, while the exclusive root belongs to the unit circle. Then the relation

$$\sup_n\{\sup_{p\in M} |U_n(P)|\} < +\infty.$$

is valid.

The proof of the theorem rests on the splitting of a difference equation like (14) and on Lemma 1.

A Priori Estimate for the Error of an Approximate Solution

For the error $z_k = u(t_k) - v_k$ we have:

$$\frac{\frac{11}{6} z_k - 3z_{k-1} + \frac{3}{2} z_{k-2} - \frac{1}{3} z_{k-3}}{\tau} + Az_k = r_k(\tau), \quad k = 5,\ldots,n, \tag{16}$$

where $r_k(\tau) = -\left(\tau^3 R_k(\tau, u) + \tilde{R}_k(\tau)\right)$.

Taking **Error! Reference source not found.** into account, we conclude that if the solution of problem **Error! Reference source not found.**, **Error! Reference source not found.** is a sufficiently smooth function, then $\| r_k(\tau) \| = O(\tau^3)$.

The following statement is true.

Theorem 4. Let A be a self-adjoint positive-definite operator in H . Then the estimate

$$\| z_{k+2} \| \le c \left(\| z_2 \| + \| z_3 \| + \| z_4 \| + \tau \sum_{i=3}^{k} \| r_{i+2}(\tau) \| \right), \tag{17}$$

is valid, where $c = const > 0$, $k = 3,\ldots,n-2$.

Let indicate the main steps of the proof of the theorem. From **Error! Reference source not found.**, we have

$$z_{k+1} = \frac{18}{11} Lz_k - \frac{9}{11} Lz_{k-1} + \frac{2}{11} Lz_{k-2} + \frac{6}{11} \tau L r_{k+1}(\tau), \tag{18}$$

where

$$L = \left(I + \frac{6}{11} \tau A \right)^{-1}.$$

If we introduce the notation

$$L_1 = \frac{18}{11} L, \quad L_2 = -\frac{9}{11} L, \quad L_3 = \frac{2}{11} L, \quad g_{k+1} = \frac{6}{11} \tau L r_{k+1}(\tau),$$

then **Error! Reference source not found.** takes the form

$$z_{k+1} = L_1 z_k + L_2 z_{k-1} + L_3 z_{k-2} + g_{k+1}.$$

Hence, by induction, we obtain (see [7], p. 68)

$$z_{k+2} = U_{k-2} z_4 + (L_2 U_{k-3} + L_3 U_{k-4}) z_3 + L_3 U_{k-3} z_2 + \sum_{i=3}^{k} U_{k-i} g_{i+2}, \tag{19}$$

where $U_k(L_1, L_2, L_3)$ are the operator polynomials which are defined by the recurrent relation

$$U_k(L_1, L_2, L_3) = L_1 U_{k-1} + L_2 U_{k-2} + L_3 U_{k-3},$$

$$U_0 = I, \quad U_{-1} = U_{-2} = 0.$$

Let us consider the characteristic equation associated with the difference equation (18)

$$\lambda^3 - \frac{18}{11}x\lambda^2 + \frac{9}{11}x\lambda - \frac{2}{11}x = 0, \tag{20}$$

where $x \in Sp(L) \subset [0,1]$ ($Sp(L)$ is the spectrum of the operator L).

It is not difficult to prove that for any $x \in [0,1]$, the real root of equation **Error! Reference source not found.** is in the unit circle, while the other two roots are complex-conjugate and belong to one and the same circle lying within the unit circle. Then by virtue of Theorem 3 the operator polynomials $U_k(L_1, L_2, L_3)$ are uniformly bounded.

Note that for proofing this result we used the well-known fact that when the argument is a self-adjoint bounded operator, the norm of the operator polynomial is equal to the C - norm of the respective scalar polynomial on the spectrum of tis operator (see e.g. [8], Ch. IX, §5).

Further note that for sufficiently smooth initial data we easily prove the estimates

$$\| u(t_k) - v_k \| = O(\tau^3), \quad k = 3,4. \tag{21}$$

Finally, **Error! Reference source not found.** and **Error! Reference source not found.** give rise to the following statement.

Theorem 5. Let A be a self-adjoint positive-definite operator in H and the solution of problem (1), (2) be a sufficiently smooth function. Then, for $\| u(t_k) - v_k \| = O(\tau^3)$, $k = 1,2$, the estimate

$$\| u(t_k) - v_k \| = O(\tau^3), \qquad k = 3,\ldots,n.$$

is true.

მათემატიკა

აბსტრაქტული ევოლუციური განტოლებისათვის ოთხშრიანი ნახევრად დისკრეტული სქემის ორშრიან სქემებზე დაყვანა და მიახლოებითი ამონახსნის ცდომილების შეფასება ასოცირებული პოლინომების გამოყენებით

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***საქართველოს ტექნიკური უნივერსიტეტის გამოთვლითი მათემატიკის დეპარტამენტი, საქართველო, თბილისი*

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

ნაშრომში, შეშფოთებათა ალგორითმის გამოყენებით, აბსტრაქტული ევოლუციური განტოლებისათვის წმინდად არაცხადი ოთხშრიანი ნახევრად დისკრეტული სქემა დაყვანილია ორშრიან სქემებზე. ამ სქემების საფუძველზე, აგებულია საწყისი ამოცანის მიახლოებითი ამონახსნი. ასოცირებული პოლინომების გამოყენებით, ჰილბერტის სივრცეში მიღებულია მიახლოებითი ამონახსნის ცდომილების შეფასება.

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