Mathematics

On Equilibrium Points of Three Point Charges

Grigori Giorgadze* **and Giorgi Khimshiashvili****

*Institute of Cybernetics, Georgian Technical University, Tbilisi, Georgia
**Institute for Fundamental and Interdisciplinary Mathematical Research, Ilia State University, Tbilisi, Georgia

(Presented by Academy Member Revaz Gamkrelidze)

We present several generalizations of our recent results on the electrostatic interpretation of points in the plane with respect to a given non-degenerate triangle *T*. First, we extend the definition of the so-called stationary charges in such a way that their existence and uniqueness hold for all points in the complement of three straight lines defined by the sides of *T*. Next, we show that, for any point *P* outside of *T*, stationary charges cannot have the same sign, and describe possible combinations of signs. For a regular triangle *T* and point *P* outside of *T*, it is also shown that the stationary charges of *P* have exactly two saddle-points and this defines a differentiable involution in the complement of *T*. The main results are complemented by a few typical examples and several related conjectures.

Point charge, Coulomb potential, critical point, equilibrium configuration, stable equilibrium, Hessian matrix, hessian, incenter of triangle

We present new applications of an approach to Maxwell's conjecture on equilibria of point charges (see, e.g., [1-3]) developed in [4, 5]. Recall that Maxwell's conjecture on the number of equilibrium points of several point charges remains unproven even in the case of three charges. A novel approach to this conjecture in the case of three charges has been developed in [5] following some ideas of [4]. In this paper, we use the same approach and generalize some results of [5] and [6]. The main ingredient of our approach is a representation of given point as an equilibrium point of certain point charges with Coulomb interaction placed at the vertices of a given triangle.

As was proven in [4], for a generic configuration of points in Euclidean plane, there exist values and positions of point charges such that the given points are stationary points of this system of point charges. Results of such type are often referred to as the *electrostatic interpretation of configurations of points* [7]. In the case of logarithmic potential, the latter topic is closely related to the theory of orthogonal polynomials and has important applications in mathematical physics (see, e.g., [7]). It should be noted that the logarithmic potential is not relevant to the problems concerned with the mathematical models of so-called *electromagnetic ion traps* constructed by W. Paul [8] which play important role in several topics of modern physics. The approach suggested in [4] and further developed in [5] was motivated by those physical problems and enabled us to construct a model of triangular electrostatic ion trap [5] similar to the quadruple ion trap of W.Paul [8].

In this paper, we apply the same approach to certain aspects of Maxwell's conjecture for three point charges. In particular, we extend the main construction of so-called *stationary charges* of point with respect to a given triangle in such a way that it includes charges of different signs. Special attention is given to the cases of regular triangle studied in [2] and three point charges with equal magnitudes considered in [3]. Some of the computations used in this paper were performed using Maple. It should be noted that all computations assisted by computer involved only symbolic and exact integer computations. Therefore, all the results in the sequel are in fact rigorously proven.

In the sequel we are concerned with the case of three point charges. To make the exposition self-contained we begin with presenting the relevant concepts in this case. Let T be a non-degenerate triangle equal to the convex hull of a triple of points $A=(A_1,A_2,A_3)$ in Euclidean plane. As usual non-degenerate means that the given three points do not belong to the same straight line. Then they define three lines l_i passing through points A_j,A_k, where (i,j,k) is a cyclic permutation of *(1, 2, 3)*. The union of these lines is denoted by L. We also denote by A^C the complement to three points A_1,A_2,A_3 in the plane and by L^C the complement to L.

Using an isometry of the plane, without loss of generality we may assume that

$$A_1=(-a,0),\ \ A_2=(a,0),\ \ A_3=(z,w)$$

with positive real numbers a and w. For a non-trivial (non-zero) triple of real numbers $Q=(q_1,q_2,q_3)$, we introduce a function E of the point $P(x,y)$ in A^C given by

$$E(P)=\frac{q_1}{\sqrt{(x+a)^2+y^2}}+\frac{q_2}{\sqrt{(x-a)^2+y^2}}+\frac{q_3}{\sqrt{(x-z)^2+(y-w)^2}}. \tag{1}$$

As is well known, this function is equal to the electrostatic potential of point charges q_i placed at vertices A_i of T [1]. Following [5], function (1) will be denoted by $E(Q@A)$. Clearly, this function is infinitely differentiable in A^C. So we can speak of its gradient $V=\mathrm{grad}E$, hessian determinant $h=h_E$ and stationary (critical) points in A^C. As usual the stationary points of E are called *equilibrium points* of system of point charges $Q@A$.

Following [5], given a point $P(x,y)$ in A^C let us search for a non-trivial triple of real numbers $Q=(q_1,q_2,q_3)$ such that P is a stationary point of the function $E(Q@A)$. Computing the gradient of $E(Q@A)$ and setting it equal to zero we get a system of two linear equations for unknowns q_1,q_2,q_3 :

$$\begin{cases} \dfrac{q_1(x+a)}{((x+a)^2+y^2)^{3/2}}+\dfrac{q_2(x-a)}{((x-a)^2+y^2)^{3/2}}+\dfrac{q_3(x-z)}{((x-z)^2+(y-w)^2)^{3/2}}=0, \\ \dfrac{q_1 y}{((x+a)^2+y^2)^{3/2}}+\dfrac{q_2 y}{((x-a)^2+y^2)^{3/2}}+\dfrac{q_3(y-w)}{((x-z)^2+(y-w)^2)^{3/2}}=0. \end{cases} \tag{2}$$

For a given point $P(x,y)\in A^C$, real numbers q_1,q_2,q_3 are called the *normalized stationary charges* of P with respect to T if they satisfy the above two linear equations and the normalizing condition $q_1+q_2+q_3=1$.

This is an extension of definition used in [5], where the stationary charges were only introduced for points inside *T*. By Theorem 1 in [5], for each point *P* inside *T,* there exists a uniquely defined triple of positive normalized stationary charges. Explicit formulas for these charges are also given in [5].

In this paper, we use *another form of normalizing condition*. Namely, we assume that the charge q_3 is always equal to one. Then a slight modification of the proof given in [5] yields that any point *P* in A^C has a uniquely defined system of stationary charges with the above normalizing condition. This system of charges will be denoted $Q(P;T)$. An important difference is that now the stationary charges may have different signs. The possible combinations of signs will be described in the sequel.

Solving the system (2) with $q_3 = 1$ we get that the stationary charges are equal to

$$r = \frac{\left(a^2 + 2ax + x^2 + y^2\right)^{3/2}\left(aw - ay - wx + yz\right)}{2ay\left(w^2 - 2wy + x^2 - 2xz + y^2 + z^2\right)^{3/2}}, \tag{3}$$

$$s = \frac{\left(a^2 - 2ax + x^2 + y^2\right)^{3/2}\left(aw - ay + wx - yz\right)}{2ay\left(w^2 - 2wy + x^2 - 2xz + y^2 + z^2\right)^{3/2}}. \tag{4}$$

The following theorem generalizes some results of [5] and [6].

Theorem 1. Any point P in A^C has a uniquely defined system of normalized stationary charges $Q(P;T)$. All stationary charges are positive if and only if the point P lies inside T. One of the stationary charges vanishes if and only if point P belongs to $L \setminus A$. The combination of signs of stationary charges remains the same for each connected component of the complement L^C.

The proof relies on interpretation of factors in formulas (3), (4) as powers of distances and oriented areas of arising triangles ΔPA_iA_j. It is then easy to see that the sign of second factor in the numerators of (3), (4) remains the same for each component U_j of the complement L^C. Obviously, one of those oriented areas vanishes if and only if point *P* lies in *L*. In the latter case the situation fits a model of linear electrostatic ion trap considered in [6] and the above formulas for stationary charges coincide with the formulas given in [6].

By Theorem 1, to determine the combinations of signs of stationary charges it is sufficient to compute it for just one point in each of the seven connected components of L^C. Let us illustrate this situation by an example which will also be used in further considerations.

Example 1. Let us consider a triple of points $\{(-1,0),(1,0),(0,1)\}$ forming an isosceles right triangle *T.* Its sides lie on the lines

$$l_1 = \{y = 0\},\ l_2 = \{x + y - 1 = 0\},\ l_3 = \{y - x - 1 = 0\}.$$

Let us numerate the six unbounded components U_j of L^C counterclockwise starting with the first quadrant. Thus U_1 is defined by inequalities

$$\{y > 0, x + y - 1 > 0, x - y + 1 > 0\},$$

and other components are defined by similar linear inequalities. For $i, j = 1,...,6,$ domains U_i and U_j are called *conjugate* (with respect to *T*) if $|i-j|=3$. For example, $U_4 = \{y<0, y-x-1<0\}$ is conjugate to U_1. The interior of T is denoted by U_7 and we already know from [5] that the stationary charges are all positive in U_7. Let us take points $P=(1,1)$ and $P=(-2,-1/2)$ lying in conjugate domains U_1 and U_4 respectively. From formulas (3, 4) we get $Q(P;T)=\left(-5\sqrt{5}/2,1/2,1\right)$ and $Q(R;T)=\left(-7\sqrt{5}/50,37\sqrt{37}/250,1\right)$. So we see that the combinations of signs of stationary charges coincide for U_1 and U_4. Analogously, it is easy to verify that the combinations of signs of stationary charges coincide in pairs (U_2,U_5) and (U_3,U_6), i.e. in each pair of conjugate components of L^C.

Remark 1. Clearly, similar notations and definitions are applicable for any non-degenerate triangle. It can be proven that the mentioned coincidence of combinations of signs in pairs of conjugate components of L^C holds for any non-degenerate triangle *T* but we do not discuss the general result for the reason of space.

In line with approach of [5] it is now natural to wonder which types of equilibrium points of stationary charges arise in the connected components of L^C. This issue was discussed in [5] for the interior domain U_7 of an isosceles triangle *T*. We complement the discussion in [5] by solving the problem for all points outside a regular triangle *T*.

To this end, we use the hessian h_P of stationary charges introduced in [5]. According to [5] a non-degenerate stationary point *P* is a non-degenerate minimum point of its stationary charges $Q(P;T)$ if and only if $h_P(P)>0$. If $h_P(P)<0$, point *P* is a non-degenerate saddle point of its stationary charges of Morse index one. Recall that according to [5] there exists a convex domain $S(T)$ inside a regular triangle *T* such that, for any point *P* in $S(T)$, one has $h_P(P)>0$ and so the point *P* is a minimum point of its stationary charges. This result was used in [5] to suggest a scenario of Coulomb control in $S(T)$ analogous to the scenarios discussed in [9]. The following result shows that the situation is essentially different for points outside *T*.

Theorem 2. For any point *P* outside a regular triangle *T*, stationary charges of *P* with respect to *T* have exactly two equilibrium points which are non-degenerate saddle points: the point *P* itself and a certain point P^***.** The point P^***,** called the *T*-conjugate of *P*, lies in the conjugate component of the component of L^C containing point *P*.

The proof uses Theorem 1 and some results of [2]. As was shown in [2], for any values q_i of vertex charges with $q_1q_2q_3<0$, there exist only two stationary points of the corresponding Coulomb potential (1). By Morse equality both such stationary points are saddle points. Let us denote by *D* the mapping defined on L^C, which sends each point *P* to its T-conjugate P^*. Consider the image $D(U_j)$, where U_j is the component of L^C containing point *P*. By implicit differentiation of relations (3), (4) it is easy to verify that mapping *D* is differentiable and locally one-to-one. Hence the set $D(U_j)$ is connected as the continuous image of connected component U_j. From Theorem 1 follows that $D(U_j)$ should lie either in U_j itself or in its conjugate component U_j. To find out which of these two possibilities is realized it is sufficient to compute the conjugate of just one point P_j in U_j. Direct computation shows that $D(P_j)$ lies in the conjugate component of U_j, which completes the proof.

Remark 2. From the local invertibility of D follows that D is a diffeomorphism between a component U_j and its conjugate component U_j. Moreover, from the definition of D follows that the composition $D \circ D$ is equal to the identity mapping of L^C. Thus we have constructed a natural differentiable involution acting in the set L^C. An interesting problem is to investigate its behavior near the boundaries of components U_j. It is easy to show that boundary points, i.e. the points lying in $L \setminus A$, are degenerate stationary points of their stationary charges with respect to *T*. So the behavior of *D* near such points should exhibit some kind of bifurcation and it is interesting to describe its topological type and normal form. As a first step one could investigate the behavior of *D* near each vertex A_i. Computations show that the boundaries of two conjugate domains are mapped to each other in one-to-one manner and the situation has certain similarity with the boundary behavior of conformal mappings although it can be shown that *D* is not conformal.

To illustrate Theorem 2 by we present some computations for a concrete regular triangle.

Example 2. Let *T* be a regular triangle with vertices $(-1,0),(1,0),(0,\sqrt{3})$ and let $P=(0,2)$ be a point in component U_2 of L^C. Stationary charges $Q(P;T)$ in this case are $(-5\sqrt{5}/4,-5\sqrt{5}/4,1)$. From the fact that there exist only two stationary points of $Q(P;T)$, and the symmetry with respect to Oy -axis follows that the T-conjugate $D(P)$ also lies on Oy -axis. Hence it is sufficient to find the ordinate y of P^* which, as can be easily seen should be a root of an algebraic equation:

$$\left(y^2+1\right)^3-4y^2\left(y-\sqrt{3}\right)^4=0. \tag{5}$$

Using Sturm's algorithm it is easy to verify that this equation has only two real roots. Solving it we get that the values of roots: *2* (exact) and –0.1414 (approximate). So the second stationary point is $P^*=(0,-0.1414)$. Computing the values of hessian h_P at these stationary points we get $h_P(P)=-1/5$, $h_P(P^*)=-2587328/7226562$. So both stationary points are non-degenerate saddles as was expected. Performing the same computations for point $P=(0,4)$ we get

$$Q(P;T)=\left(-51\sqrt{17}/216,-51\sqrt{17}/216,1\right),\ P^*=(0,-0.3375).$$

Thus we see that as the ordinate of point *P* grows the ordinate of point P^* decreases. In fact, this behavior of conjugate point on Oy -axis can be rigorously proven. The same type of behavior is observed on other lines passing through vertices of triangle *T*, which can be used for investigating the boundary behavior of mapping *D* near the vertices of *T*.

Remark 3. Using the same arguments as above one can obtain an analog of Theorem 2 in the case of an isosceles right triangle studied in [2].

We notice that in the above examples the stationary charges of different signs have only two stationary points which are non-degenerate saddles. So it is natural to wonder if three charges of different signs always have only two equilibrium points. This question seems to remain unanswered in the literature and we were unable to solve this problem in general. However it can be answered positively in some special cases one of which is given below.

Theorem 3. *Three point charges of equal magnitudes with different signs at vertices of any non-degenerate triangle have always two equilibrium points which are non-degenerate saddles.*

Since by [3] the number of equilibrium points of non-equal charges in this case is two, this can be proven using the same reasoning as in the proof of Theorem 2. This result gives some evidence to conjecture that three point charges of different signs always have only two equilibrium points, which would prove an important special case of Maxwells' conjecture for three point charges.

მათემატიკა

სამი წერტილოვანი მუხტის წონასწორობის წერტილების შესახებ

გ. გიორგაძე* და გ. ხიმშიაშვილი**

**საქართველოს ტექნიკური უნივერსიტეტი, კიბერნეტიკის ინსტიტუტი, თბილისი, საქართველო*
***ილიას სახელმწიფო უნივერსიტეტი, ფუნდამენტური და ინტერდისციპლინარული მათემატიკური კვლევების ინსტიტუტი, თბილისი, საქართველო*

(წარმოდგენილია აკადემიის წევრის რ. გამყრელიძის მიერ)

სტატიაში განხილულია ავტორების ბოლოდროინდელი შედეგების განზოგადება მოცემული გადაუგვარებელი T სამკუთხედის მიმართ სიბრტყეზე წერტილების კონფიგურაციის ელექტროსტატიკური ინტერპრეტაციის შესახებ. განზოგადებულია ე. წ. სტაციონარული მუხტის ცნება იმგვარად, რომ მისი არსებობა და ერთადერთობა გამომდინარეობს ყველა წერტილისთვის სამკუთხედის გვერდებისგან შედგენილი წრფეების დამატებიდან. ამის შემდეგ ნაჩვენებია, რომ T-ს ყოველი P გარე წერტილისათვის სტაციონარულ მუხტებს არ შეიძლება ჰქონდეთ ერთნაირი ნიშანი. აღწერილია მათი შესაძლო კომბინაციები. ტოლგვერდა T სამკუთხედისათვის და მის გარეთ აღებული P წერტილისათვის ნაჩვენებია, რომ სტაციონარული მუხტები უნაგირის ტიპის წერტილებია, რომელთა რაოდენობაა 2. წერტილთა ეს წყვილები განსაზღვრავენ T-ს დამატებაზე დიფერენცირებად ინვოლუციას. ძირითად შედეგებთან ერთად მოყვანილია რამდენიმე ტიპური მაგალითი და ამ შედეგებიდან გამომდინარე ჰიპოთეზები.

REFERENCES

1. Gabrielov A., Novikov D., Shapiro B. (2007) Mystery of point charges. *Proc. Lond. Math. Soc.,* **95**, 2:443-472.
2. Tsai Y.-L. (2011) Special cases of three point charges. *Nonlinearity,* **24,** 12:3299-3321.
3. Tsai Y.-L. (2015) Maxwell's conjecture on three point charges with equal magnitudes. *Physica D* **309:** 86-98.
4. Khimshiashvili G. (2016) Configurations of points as Coulomb equilibria. *Bull. Georgian Natl. Acad. Sci.,* **10,** 1:20-27.
5. Giorgadze G., Khimshiashvili G. (2021) Triangles and electrostatic ion traps. *J. Math. Phys.* **62,** 5, 053501: 10 pp.
6. Giorgadze G., Khimshiashvili G. (2018) Equilibria of point charges in a line segment. *Proc. I.Vekua Inst. Appl. Math.* **68**: 16-27.
7. Simanek B. (2016) An electrostatic interpretation of the zeros of paraorthogonal polynomials on the unit circle. *SIAM J. Math. Anal.* **48,** 3:2250-2268.
8. Paul W. (1990) Electromagnetic traps for charged and neutral particles. *Rev. Mod. Phys.* **62:** 531-540.
9. Khimshiashvili G., Panina G., Siersma D. (2014) Coulomb control of polygonal linkages. *J. Dynam. Control Syst.* **20,** 4:491–501.

Received June, 2021