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Mathematics

On Random Planar Endomorphisms

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Random polynomials with independent identically distributed Gaussian coefficients are considered and explicit estimates for the expected number of real roots are given. It is found that standard random polynomial planar endomorphism is proper. It is also shown that the dehomogenized polynomial is a random Gaussian polynomial. *© 2021 Bull. Georg. Natl. Acad. Sci.*

Random polynomial endomorphism, Gaussian distribution, proper mappings, homogeneous polynomial

In this paper we consider random endomrphisms of the plane defined by two independent random polynomials in two variables. For random polynomials in one variable, M. Kac found the expected number of real roots for polynomial of fixed algebraic degree d with independent standard Gaussian random variables as coefficients [1]. We call this case the *Kac setting.* Today the expected number of real roots is known for many classes of random polynomials.

Similar problems were studied for polynomials in several variables and random polynomial mappings defined by systems of such polynomials. M. Shub and S. Smale computed the expected number of real roots in arbitrary dimension for certain special distribution of coefficients [2]. Their results were further generalized in [3].

As was suggested in [4], similar problems can be studied for topological invariants of random polynomial mappings. They are already quite interesting for random planar endomorphisms which we call *random plends*. Notice that results in [4, 5] refer to random plends with rotation invariant Gaussian distribution of coefficients introduced in [2]. G. Khimshiashvili computed the average gradient degree in *Shub-Smale setting* in arbitrary dimension. These results can be used for estimating many other topological invariants of such random polynomials and mappings in any dimension. However this important progress in the topological study random polynomials was only achieved in Shub-Smale setting.

For many reasons it is also interesting to compute or estimate the topological degree of a Gaussian random plend with *independent identically distribution* (iid) coefficients i.e., in Kac setting. The Kac setting appeared more difficult than the Shub-Smale setting and up to our knowledge there exists no simple formulae for the expected number of real roots and average gradient degree in Kac setting. We give some

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results in this case and explain how our results can be generalized for other i.i.d. coefficients (i.e., not necessarily standard normal) using the results from [3] .

Recall that if ξ is a random variable with Gaussian (normal) density

$$f_\xi = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

with $\sigma > 0, -\infty < a < +\infty$ then parameters α and σ determine its expectation and variance: $a = E\xi$, $\sigma^2 = D\xi$.

In other words, a normal distribution is defined by

$$P(\xi \in B) = \Phi_{a,\sigma^2}(B) = \frac{1}{\sigma\sqrt{2\pi}} \int_B e^{-\frac{(u-a)^2}{2\sigma^2}} du .$$

If $a = 0$ and $\sigma = 1$ we obtain the standard normal distribution $\Phi_{0,1}$ (often denoted simply $\Phi(x)$)

$$\Phi(x) = \Phi_{0,1}(-\infty, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du .$$

If (ξ, η) is a pair of random variables then the number

$$\operatorname{cov}(\xi, \eta) = E\left[\left(\xi - E\xi\right)\left(\eta - E\eta\right)\right]$$

is called covariation of $a = 0$ and η . If $\operatorname{cov}(\xi, \eta) = 0$, then $a = 0$ and η are called non-correlated, in particular (stochastically) independent random variables are non-correlated. The variance $D\xi$ is defined as $\operatorname{cov}(\xi, \xi) = D\xi$.

Consider now a Gaussian random polynomial on $\mathbb{R}^2$ of algebraic degree n such that its coefficients are i.i.d. standard normals. Taking a pair of independent polynomials of such type we obtain a random plend which will be called *standard random plend* of algebraic degree n . This is precisely the type of random polynomial we will consider.

We reproduce now the necessary result about the expected number of real roots from [3]. Let

$$P(x) = a_0 + a_1 x + a_2 x + \cdots + a_n x^n$$

be a non-zero polynomial. Define the two vectors:

$$\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \text{ and } \vec{V} = \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{pmatrix} .$$

The curve in $\mathbb{R}^{n+1}$ traced by $\vec{V}(t)$ as t runs over the line is called *moment curve*.

The condition that $x = t$ is a zero of the polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is precisely the condition that $\vec{a}$ is perpendicular to $\vec{V}(t)$. For each point of p of the unit sphere S^n , let us denote by $p_\perp$ the equator of S^n orthogonal to the radius-vector of p . With this notation, $\vec{V}(t)_\perp$ is the set of polynomials which have t as a zero.

Define unit vectors $\vec{\boldsymbol{a}} \equiv \frac{\vec{a}}{\| \vec{a} \|}$ and $\vec{\gamma}(t) \equiv \frac{\vec{V}(t)}{\| \vec{V}(t) \|}$. As before, $\vec{\gamma}$ corresponds to the polynomials which have t as a zero.

If all a_i are independent standard normals, then the vector $\vec{\boldsymbol{a}}$ is uniformly distributed on the sphere S^n since the joint density function in spherical coordinates is a function on the radius alone. Thus such a random polynomial can be identified with a uniformly distributed random point on the sphere and, as is explained in [3], the expected number of its real roots E_n can be found from the "kinematic formula"

$$\frac{|\vec{\gamma}_{\perp}|}{area\ of\ S^n} = \frac{|\vec{\gamma}_{\perp}|}{\pi}$$

where $|\vec{\gamma}|$ is the length of $\vec{\gamma}$ and $|\vec{\gamma}_{\perp}|$ is the area "swept out" by the associated equator of a point moving on curve $\vec{\gamma}$. Hence

$$E_n = \frac{|\vec{\gamma}_{\perp}|}{\pi}$$

which enables one to compute E_n explicitly in terms of covariance matrix of coefficients. We reproduce the general result from [3].

If $f_0, f_1, \ldots, f_n$ is any collection of rectifiable functions, define the analogue of the moment curve

$$\vec{v}(t) = \begin{pmatrix} f_0(t) \\ f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

The function $\frac{1}{n}\|\vec{\gamma}'(t)\|$ is the density of real zeros, i.e. its integral over $\mathbb{R}$ is the expected number of real zeros. The density can also be found for a coefficients vector $\vec{a} = (a_0, a_1, \ldots, a_n)^T$ having any multivariate distribution with zero mean. If a_i are normally distributed, $E(a) = 0$ and $E(aa^T) = C$ (i.e. $\vec{a}$ has a multivariate central normal distribution with covariance matrix C) then it is easy to see that $\vec{a}$ has this distribution if and only if $C^{-1/2} \times \vec{a}$ is a vector of standard normals. Since

$$\vec{a} \times \vec{v}(t) = C^{-1/2} \times \vec{a} \times C^{1/2} \times \vec{v}(t)$$

we conclude that the density of real zeros with coefficients having an arbitrary central multivariate normal distribution is $\frac{1}{\pi}\|\vec{w}'(t)\|$, where $\vec{\omega}(t) = C^{1/2} \times \vec{v}(t)$ and

$$\vec{W}(t) \equiv \frac{\vec{\omega}(t)}{\|\vec{\omega}(t)\|}.$$

Theorem 1. ([3]) Let $\vec{v}(t) = \left(f_0(t), f_1(t), \ldots, f_n(t) \right)^T$ be any collection of differentiable functions and $a_0, a_1, \ldots, a_n$ be the elements of multivariable normal distribution with mean zero and covariance matrix *C*. The expected number of real zeros in an interval I of the equation

$$a_0 f_0(t) + a_1 f_1(t) + \cdots + a_n f_n(t) = 0$$

is

$$\int_I \frac{1}{\pi}\left\|\vec{W}(t)\right\| dt$$

which can be rewritten as

$$\frac{1}{\pi}\int_I \left(\frac{\partial^2}{\partial x \partial y}\left(\log\left(\vec{v}(x)^T \times C \times \vec{v}(y)\right)\right)\Bigg|_{y=x=1} \right)^{\frac{1}{2}} dt$$

Let $\mathbb{R}_2$ be the ring of real polynomials of two variables. For $P \in \mathbb{R}_2$, let $degP$ denote its algebraic degree, i.e. the highest order of monomials which appear in P . Any P with $degP = n$ can be written as

$$P(x, y) = \sum_{k+1=0}^{n} a_{kl} x^k y^l \ ,$$

where at least one non-vanishing a_{kl} with $k + l = n$ appears. The leader P^* is defined as the sum of monomials of highest order. Obviously it is a non-trivial binary n -form.

We suppose that $a_{kl} = a_{kl}^{(\omega)}$ are standard normal random variables and define the standard random plend

$$F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$$

taking two random polynomials as its components.

Proposition 1. A standard random plend is proper with probability one.

As we know such a plend is almost certainly stable in the sense of Whitney as well. For these ω for which $F(\omega)$ is not proper we set $DegF(\omega) = 0$ and aim at expectation $E(DegF)$ of random variable $DegF$. At present we only succeeded in the case of a random gradient plend. Notice that in this case the components of F' are identically distributed but not independent. Such random plends were earlier considered in [5].

Notice that the zero set Z of a homogeneous polynomial P^* consists of a system of lines in $\mathbb{R}^2$ passing through the origin. Their intersection with the unit circle S^1 gives a finite set of points $Y = Z \cap S^1$. These points obviously appear in pairs and those pairs are in one-to-one correspondence with the real roots of polynomial in one variable $\hat{P}$ which is obtained from P^* by dehomogenization (i.e. we divide $P^*(x, y)$ by y^n and introduce a new variable $t = \frac{x}{y}$) . In other words, the number k of points in Y equal $2r$, where r is the number of real roots of $\hat{P}$ [6].

Proposition 2. $\hat{P}$ *is a Gaussian random polynomial of algebraic degree* n *with i.i.d. standard normal coefficients.*

We conclude that one can compute the expected number of real roots $E(r)$ of $\hat{P}$ using Theorem 1. Hence the fact that $E\left(\left|DegP'\right|\right)$ has the asymptotic indicated in the theorem follows from the asymptotic Kac formula [3]. The details of this argument will be presented elsewhere for the reason of space.

We point out that the similar scheme is applicable to many other classes of random polynomials. What is needed for our argument to work, is that the number of real roots of the dehomogenized leader could be easily computed or estimated. Examples of such distributions can be found in [3].

It should be added that it remains unclear how to estimate the average topological degree of an arbitrary (not necessarily gradient) Gaussian random plend. Some results for rotation invariant Gaussian distribution were obtained in [5]. As is well known other geometrical characteristics of a random plend can be estimated in terms of its topological degree so the results obtained in the mentioned publications can be used for estimating the expectations of other invariants of random plends.

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შემთხვევითი ბრტყელი ენდომორფიზმების შესახებ

თ. ალიაშვილი

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