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Mathematics

On the Notion of the Basis of Finite Dimensional Vector Space

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(Presented by Academy Member Revaz Gamkrelidze)

In this paper we show that in the expository texts on linear Algebra, the notion of a basis could be introduced by an argument much weaker than Gauss' reduction method. Our aim is to give a short proof of a simply formulated lemma, which in fact is equivalent to the theorem on frame extension, using only a simple notion of the kernel of a linear mapping, without any reference to special results, and derive the notions of basis and dimension of a finite dimensional vector space in a quite intuitive and logically appropriate way, as well as obtain their basic properties, including a lucid proof of Steinitz's theorem. *© 2021 Bull. Georg. Natl. Acad. Sci.*

Bases of vector space, linear mapping, Gauss reduction, Steinitz's theorem

Preliminary Remarks

The structure of the basic object of Linear Algebra, the finite dimensional vector space, is amazingly simple. It is completely defined by a single notion – its basis. Therefore, different versions of Steinitz's theorem on the extension of an arbitrary frame (a sequence of linearly independent vectors) in a finite dimensional vector space to a basis by an appropriately chosen subsequence of a preassigned basis of the space were always of a special interest to the authors of numerous texts on the subject, since the method of a frame extension is the standard most frequently used path leading from axioms of a finite dimensional vector space to the notions of its basis and dimension. To describe the corresponding transition, a whole bulk of necessary definitions and propositions is traditionally introduced already before the linear mappings make their first appearance. As a result, the authors are compelled, explicitly or implicitly, to a full measure usage of Gauss' inductive procedure of a matrix reduction to the triangular form. In this respect, it is instructive to compare the expositions of the corresponding material in two standard texts on the subject, [1] and [2], published with the time lag of 50 years. For a modern introductory course refer to [3].

We consider that in most texts on linear algebra the linear mappings are introduced too late, contrary to a well known motto, according to which "morphisms in a category are at least as useful as objects are".

In this Note, we show that the notion of a basis could be introduced by an argument much weaker than Gauss' reduction method. We suggest to rearrange the exposition of the fundamental material and to introduce linear mappings and their basic properties right after the vector space axioms are listed and properly discussed. Our aim is to give a short proof of a simply formulated lemma, which in fact is equivalent to the above-mentioned theorem on frame extension, using only a simple notion of the kernel of a linear mapping, without any reference to special results, and derive the notions of basis and dimension of a finite dimensional vector space in a quite intuitive and logically appropriate way, as well as obtain their basic properties, including a lucid proof of Steinitz's theorem. This text is based on [4].

A Short List of Necessary Initial Notions

There seems to be a complete unanimity concerning the axioms for vector spaces and their linear mappings over an arbitrary field of scalars Λ. Therefore we shall assume without further comments that $\mathbf{E},\mathbf{F}$ are (not necessarily finite dimensional) vector spaces over a field Λ; L is a linear mapping from $\mathbf{E}$ to $\mathbf{F}$, (i. e. L commutes with the vector addition and with multiplication of vectors by scalars), and the action on vectors of scalars and mappings is from the left, L

$$\lambda\mu\bullet x = \lambda\bullet\mu x,\ \lambda(x+y) = \lambda x + \mu y,$$
$$L(\lambda x + \mu y) = \lambda\bullet Lx + \mu\bullet Ly,$$

$$x, y \in \mathbf{E},\ \ \lambda,\mu \in \Lambda,\ \ L \in Hom(\mathbf{E},\mathbf{F}).$$

Together with the standard general set-theoretic notions related to the mapping L,

$$domL = \mathbf{E},\ \ codomL = \mathbf{F},\ \ imL = L(\mathbf{E}),$$

it is useful to have at the disposal from the very beginning specifically linear notions of a (linear) subspace $\mathbf{E}\subset\mathbf{F}$ and the factor $\mathbf{E}/\mathbf{F}$, in particular, the notions of kernel $\ker L$ and the factor $\mathbf{E}/\ker L$.

For an adequate definition of the basis and its appropriate discussion, we should have a certain freedom in handling finite sequences of vectors in $\mathbf{E}$. An arbitrary sequence of length n of vectors in $\mathbf{E}$ will be presented as an n-row matrix,

$$|\, x_1 \,|\, x_1,\ldots,x_n \,|\, x_j \,| = \|\, x_j \,\|_n = \|\, x_j \,\|,\ \ x_j \in \mathbf{E},$$
$$j = 1,\ldots,n.$$

Every sequence of vectors could be *extended* by ascribing to it new vectors from the space. It is useful to remember that a sequence of length n of vectors in $\mathbf{E}$ is not a subset of $\mathbf{E}$, but rather a function to $\mathbf{E}$ on the ordered set of first n naturals, $im\,\|\, x_j \,\|\subset \mathbf{E}$. If $im\,\|\, x_j \,\|$ is a subset of a subspace $\mathbf{F}\subset\mathbf{E}$ we say that the sequence $\|\, x_j \,\|$ belongs to $\mathbf{F}$ and write $\|\, x_j \,\|\prec \mathbf{F}$.

The initial list of basic items should contain the following notions: linear combination of vectors of a sequence, linearly independent and dependent sequences of vectors of $\mathbf{E}$, n- frames – linearly independent sequences of length n, or $rankn$,

$$\|\, e_j \,\| = \|\, e_j \,\|_n,\ rank\,\|\, e_j \,\|_n = n.$$

The rank of an arbitrary sequence, $rank\,\|\, x_j \,\|, \|\, x_j \,\|\prec \mathbf{E}$, is defined as the maximal rank of frames contained (as subsequences) in $\|\, x_j \,\|$. The linear hull $[\|\, x_j \,\|]\subset \mathbf{E}$ of an arbitrary sequence of vectors $\|\, x_j \,\|\prec \mathbf{E}$, linear hull of an arbitrary subset $A\subset\mathbf{E}$ – minimal subspaces in $\mathbf{E}$ (with respect to the set-theoretic inclusion) containing, respectively, subsets $im\,\|\, x_j \,\|$ and A.

The Basic Lemma. Formulation and a Preliminary Discussion. A frame $\|\, e_j \,\|\prec \mathbf{F}\subset\mathbf{E}$ is *maximal in a subspace* $|\,\mathbf{F}$ if the rank of an arbitrary sequence $\|\, x_j \,\|\prec \mathbf{F}$ is bounded by the rank of $\|\, e_j \,\|$,

$$\| x_j \| \prec \mathbf{F} \Rightarrow rank \| e_j \| \geq rank \| x_j \|.$$

A frame $\| e_j \|_n \prec \mathbf{F}$ could be *extended* (is *extendable*) in $\mathbf{F}$, if there exists a vector $f \in \mathbf{F}$ such that the extended sequence $e_1,...,e_n, f$ is again a frame (of rank $n+1$). Intuitively, maximality of a frame could be considered as a "global version" of the "local property" of a frame to be non-extendable. Every maximal frame in $\mathbf{F}$ is evidently not extendable. The inversion of the assertion is also true – every non-extendable frame in $\mathbf{F}$ is maximal in $\mathbf{F}$ (the Steinitz' extension theorem), but the proof is not trivial and in fact belongs right to the core of the problem under discussion of giving proper definitions of a basis and dimension of a finite dimensional vector space. We shall formulate now a simple lemma, which easily clears up all interrelations between the notions of maximality of a frame and its ability to be "extendable", and suggests natural intuitive definitions of a basis and dimension.

The basic lemma. Every frame $\| e_j \| \prec \mathbf{E}$ is maximal in its linear hull $[\| e_j \|]$ and extendable in every subspace $\mathbf{F} \supset [\| e_j \|]$, if the inclusion is strong.

The ability of being "extendable" under the given conditions is evident – it is achieved by ascribing to $\| e_j \|$ of an arbitrary vector $f \in \mathbf{F}, f \notin [\| e_j \|]$. The maximality of $\| e_j \|$ in the linear hull $[\| e_j \|]$ is equivalent to each of the following two assertions.

1. Every frame of rank n, $\| f_j \|_n$, in the linear hull $[\| e_j \|_n]$ is not extendable there, or, equivalently, the following system of n inclusions is valid,

$$e_i \in [\| e_j \|_n], i = 1,...,n.$$

2. The rank of an arbitrary sequence in $\mathbf{E}$ consisting of linear combinations of a fixed sequence of n vectors from $\mathbf{E}$ does not exceed n.

A simple inductive proof of the system of inclusions, based on the notion of the kernel of a linear mapping, is given at the end of the Note. Before, in next two sections, we shall derive from the formulated lemma simple and intuitive definitions of a basis and dimension of a finite dimensional vector space, as well as a lucid short proof of the Steinitz' theorem on the extension of a frame.

Definition of a Basis and Dimension of a Finite Dimensional Vector Space

A vector space $\mathbf{E}$ is finite dimensional if it contains a finite set of generators – a finite subset $G \subset \mathbf{E}$ with the linear hull coinciding with $\mathbf{E}$. Since G is finite, it contains a frame $\| e_j \|_n \prec G$ (of maximal rank), which spans the whole space $\mathbf{E} = [\| e_j \|_n]$, hence, according to the lemma, the frame is maximal in $\mathbf{E}$. Thus every finite dimensional vector space contains maximal frames – bases of the space. Their common rank n is the dimension of the space, and every vector $x \in \mathbf{E}$ is uniquely represented as

$$x = \sum_1^n \lambda^\alpha e_\alpha, \ \ \lambda^j \in \Lambda, \ \ j = 1,...,n.$$

Thus, every basis of a finite dimensional vector space is an irreducible system of generators of the space.

Conversely, if a sequence $\| x_j \|_n$ in a vector space $\mathbf{E}$ is given such that every vector $x \in \mathbf{E}$ is uniquely represented as

$$x = \sum_1^n \lambda^\alpha x_\alpha, \ \ \lambda^j \in \Lambda,$$

then, according to the lemma, $\| x_j \|_n$ is a maximal frame of rank n, or a basis in $\mathbf{E}$.

The Steinitz' Theorem on the Frame Extension

In $k+l$-dimensional vector space $\mathbf{E}^{k+l}$ a basis $B = \| e_j \|_{k+l}$ and an arbitrary k-frame $\| f_j \|_k$

are given. From general considerations it easily follows by induction that B contains a subsequence $e_{i_1},...,e_{i_r}$ of a certain length $r \le k+l$ such that the frame $\| f_j \|_k$ extended by the subsequence transforms into a frame of length $k+l$

$$B^{'} = \| f_1,..., f_k; e_{i_1}...,e_{i_r} \| \prec \mathbf{E}^{k+l},$$

containing the linear hull of B , which is the whole space $\mathbf{E}^{k+l}$, hence

$$[B^{'}] = \mathbf{E}^{k+l}.$$

According to the lemma, the obtained frame $B^{'}$ is maximal in $\mathbf{E}^{k+l}$, i. e. is a basis of the $k+l$-dimensional vector space $\mathbf{E}^{k+l}$, hence $r = l$, and we have extended the initial k -frame $\| f_j \|_k$ by a subsequence of length $r = l$ of a preassigned basis B to a new basis $B^{'}$ of the space $\mathbf{E}^{k+l}$.

Proof of the Basic Lemma

We shall prove the system of inclusions (1) by induction performing it on the rank n of the frame $\|e_j\|_n$. The asseretion is evident for $n=1$ since the linear hull in this case coincides with the family of vectors

$$[\| e_j \|]_1 = \{\lambda e_1 \mid \lambda \in \Lambda\}.$$

Assuming that the assertion is proved for all natural $k \le n-1$, consider the case $k = n.$ Introduce n linear mappings

$$L_i \in Hom([\| e_j \|]_n, [\| f_1,..., \widehat{f_i},..., f_n \|]),$$
$$i = 1,...,n,$$

by defining L_i on vectors $e_1,...,e_n$ according to the equations

$$L_i e_j = f_j, \ \ i \ne j, \ L_i e_i = 0, \ \ i, j = 1,...,n.$$

The kernel of L_i is the subspace

$$\ker L_i = \{\lambda e_i \mid \lambda \in \Lambda\},$$

and the image of L_i – the linear hull

$$imL_i = [\| f_1,..., \widehat{f_i},..., f_n \|].$$

The restriction of L_i on the subspace $[\| f_j \|_n] \subset [\| e_j \|_n]$ has a nonzero kernel since otherwise the n-frame $\| L_i f_j \|, j = 1,...,n$ would be embedded in the linear hull of $(n-1)$-frame $\| f_1,..., \widehat{f_i},..., f_n \|$, which contradicts the inductive assumption. Hence,

$$\lambda e_i \in \ker L_i |_{[\| f_j \|_n]}, \lambda \in \Lambda, i = 1,...,n,$$

or

$$e_i \in [\| f_j \|_n], \forall i = 1,...,n.$$

This proves the Lemma.

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REFERENCES

1. Schreier G., Sperner E. (1931) Einführung in die Analytische Geometrie und Algebra, Bd. B.G. Teubner, Leipzig.
2. Deheuvels R. (1981) Formes quadratiques et groupes classiques. Presses Universitaires de France.
3. Axler Sh. (2015) Linear Algebra Done Right. Springer Verlag.
4. Gamkrelidze A., Giorgadze G. (2017) On the notion of a basis of a finite dimensional vector space, arXiv:1712.04244.

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