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BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 16, no. 1, 2022

Informatics

Mathematical Model of Urban Planning for Sustainable Development and Reconstruction of the City

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The tasks of urban planning are currently being solved by the traditional method on interactive maps, where zoning and input of relevant data are carried out manually. The solution to this issue using mathematical modeling is known as the so-called, Voronoi diagram. Practical application of this method has not been widely used. The point is that the objects that determine the land use of the region to be reviewed, are considered as homogeneous taking into account the fact that they have the same potential, which is far from reality. In this paper, each project object is different and they have different potentials that makes it possible to identify „inconsistencies" in urban planning parameters both – in terms of urban planning and in terms of socio-economic issues. Appropriate application software system is made based on the above mentioned method as well. *© 2022 Bull. Georg. Natl. Acad. Sci.*

Urban system, modeling, Voronoi diagram

The optimal solution of urban design for sustainable development and urban reconstruction is an urgent problem. Urban planning and zoning tasks are currently being solved with the help of interactive maps where related technical data are provided manually.

The solution of this problem with the help of mathematical modeling is known as the Voronoi diagram [1]. after the development of an appropriate algorithm, this method could become the basis for the development of an application software package. But the practical application of this method has not been widely used for many reasons. One of the main points is that each of the objects that determines the planning of the territory is considered as homogeneous and has the same potential, which is far from the reality. The method we have developed makes it possible to overcome these problems, in particular, to identify urban „inconsistencies", both in terms of urban planning and socio-economic issues, and to develop an appropriate application of software system.

Suppose that in the considered area E on the coordinate plane there are N material points (objects) $A_i(x_i;y_i),\ i\in\left[\overline{1;N}\right]$. According to the definition of Voronoi diagram, for each $i\in\left[\overline{1;N}\right]$, the „coverage zone“ of point $A_i(x_i;y_i),$ is the set of all points E, from which the distance to the point $A_i(x_i;y_i),$ is less than the distance of the other $A_j(x_j;y_j),$ (for each $j\in\left[\overline{1;N}\right],\ \ j\neq i$).

In this case the allocation of „coverage zones“ is not difficult. The „coverage zone“ of point $A_j(x_j;y_j),$ is the intersection $E\cap\left(\bigcup_{\substack{j=1\\ j\neq i}}^{n} s_{ij}\right)$, where the set S_{ij} is defined as: Median of the segment connecting point $A_j(x_j;y_j),$ and $A_j(x_j;y_j),$ divides the plane into two half-planes. S_{ij} denotes one of these two half-planes, in which point $A_j(x_j;y_j),$ is located.

Naturally, in real situations, different points (objects) can have different potentials. Therefore, the answer to the following question is undoubtedly interesting: how to build „coverage zones“ (how to build Voronoi diagram), when different points (objects) can have different potentials (weights).

Suppose that p_i denotes the potential (weight) of the point $A_i(x_i;y_i)$. If, for some point M of the set E, the condition $\frac{|MA_i|}{|MAj|}<\frac{p_i}{p_j},i,j\in\left[\overline{1;N}\right],i\neq j,$ then we say, that the influence of the material point $A_j(x_j;y_j)$ on the point – M is greater, than the influence of the point $A_j(x_j;y_j)$. If we denote by the symbol E_{ij} the sum of all points of the set $E,$ in which the influence of the point A_i is greater, than the influence of the point A_i, then, naturally, the „coverage zone“ of the point A_i will be the set: $E_i=\bigcup_{\substack{j=1,\\ j\neq i}}^{N} E_{ij}$.

As mentioned above, the construction of the Voronoi diagram means dividing the referred area into „coverage zones“ of the given material points. We can easily solve this problem if for every $i,j\in\left[\overline{1;N}\right],i\neq j,$ we can find lines that satisfy the condition $l_{ij=}\left\{M\in E;\frac{|MA_i|}{|MAj|}=\frac{p_i}{p_j}\right\}$ (such lines are called boundary lines).

To simplify it, we introduce the notation: $k_{ij}=\frac{p_i}{p_j}$.

It is easy to see that if $k_{ij}=1$, then l_{ij} is the median of the section $\left[A_i;A_j\right]$. Therefore, consider the case when $k_{ij}\neq 1$.

Without loss of generality, we may assume that $k_{ij}>1$. If $M(x;y)$ is a point of the circle l_{ij}, then, by definition, we have the equation:

$$|MA_i|=k_{ij}\cdot|MA_j. \qquad (1)$$

(1) The equation is equivalent to the following equation (based on the formula for calculating the distance between two points):

$$(x-x_i)^2+(y-y_i)^2=k_{ij}^2\cdot\left((x-x_j)^2+(y-y_j)^2\right).$$

After simple transformations we get

$$\left(x-\frac{k_{ij}^2\cdot x_j=x_i}{k_{ij}^2-1}\right)^2+\left(y-\frac{k_{ij}^2\cdot y_j=y_i}{k_{ij}^2-1}\right)^2=$$
$$=\frac{\left(x_i^2+y_i^2-k_{ij}^2\cdot x_j^2-k_{ij}^2\cdot y_j^2\right)\cdot\left(k_{ij}^2-1\right)+\left(k_{ij}^2\cdot x_j-x_i\right)^2+\left(k_{ij}^2\cdot y_j-y_i\right)^2}{\left(k_{ij}^2-1\right)^2}.$$

The numerator to the right of the obtained equation will then look like this:

$$k_{ij}^2\cdot\left((x_i-x_j)^2+(y_i-y_j)^2\right)=k_{ij}^2\cdot a_{ij}^2,$$

where a_{ij} denotes the distance between points $A_i(x_i;y_i)$ and $A_j(x_j;y_j)$.

Therefore, we get that l_{ij}, when $k_{ij}>1$, is a circle whose equation is given by:

$$\left(x-\frac{k_{ij}^2\cdot x_j=x_i}{k_{ij}^2-1}\right)^2+\left(y-\frac{k_{ij}^2\cdot y_j=y_i}{k_{ij}^2-1}\right)^2=\frac{k_{ij}^2\cdot a_{ij}^2}{\left(k_{ij}^2-1\right)^2}.$$

It is easy to show, that point: $\left(\frac{k_{ij}^2\cdot x_j=x_i}{k_{ij}^2-1};\frac{k_{ij}^2\cdot y_j=y_i}{k_{ij}^2-1}\right)$, which is the center of the obtained circle, is located on the line connecting the points A_i and A_j.

In fact, the following theorem holds:

Theorem: If the distance between two points A and B of the plane is equal to a and $k>1$ is equal to any number, then the set of all points of the plane that are at a distance k from point A is a circle, whose radius is $\frac{ka}{k^2-1}$ and whose center is located on the $(A;B)$ ray at a distance $\frac{k^2a}{k^2-1}$, from point A.

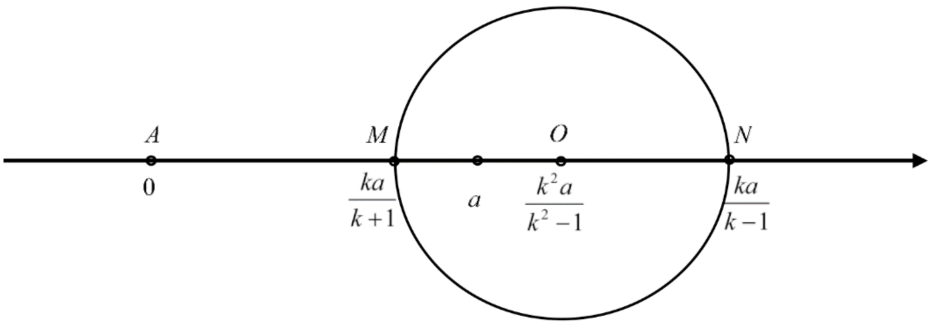

Fig. 1. Illustration of Voronoi diagram for points of different potentials.

Our result allows us to create a Voronoi-type algorithm in the case when the potentials of different objects are different and when the potential of each object P represents a vector value $P(p_1,p_2,...,p_n)$. In this case, the numerical value of the potential of the object is determined by the sum of the components of the characteristic vector of the object. Therefore, we can consider each object as a simplex, and the region to be reviewed as a simplicial complex. That allows to study the relationship of objects in the referred region based on the q-analysis method [2] and also to create an interactive applications software package for optimal zoning and building economic urban infrastructure.

ინფორმატიკა

მათემატიკური მოდელი ქალაქის ურბანული სისტემის მდგრადი განვითარებისა და რეკონსტრუქციისთვის

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ურბანული სისტემის დაგეგმვის ამოცანების გადაჭრა ტრადიციული მეთოდოლოგიით ხდება ინტერაქტიულ რუკებზე. ზონირება და შესაბამისი მონაცემების შეყვანა ხდება ხელით. აღნიშნული საკითხის გადაწყვეტა მათემატიკური მოდელირების საშუალებით, ცნობილია ე. წ. ვორონოის დიაგრამის სახით. ამ მეთოდმა პრაქტიკული გამოყენება ვერ პოვა. საქმე ისაა, რომ ობიექტები, რომლებიც განსაზღვრავენ განსახილველი რეგიონის მიწათმოწყობას, განიხილებიან როგორც ერთგვაროვანნი, იმ გაგებით, რომ აქვთ ერთნაირი პოტენციალი, რაც რეალობისაგან შორსაა. წარმოდგენილ ნაშრომში ყოველი საპროექტო ობიექტი არის განსხვავებული და მათ აქვთ სხვადასხვა პოტენციალი. რაც საშუალებას გვაძლევს დავადგინოთ ურბანული პარამეტრების „შეუსაბამობები" – როგორც ურბანული დიზაინის ისე სოციალური ეკონომიკური საკითხების მიმართ. აღნიშნული მეთოდის ბაზაზე შექმნილია გამოყენებითი პროგრამული სისტემა.

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Received January, 2022