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Mathematics

Numerical Solution of Anti-Plane Problems of the Elasticity Theory for Composite Isotropic Plane Slackened by Linear Crack

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In the present paper, the issues of the approximate solution of singular integral equation and pair systems of integral equations containing fixed-singularity are studied. The studied integral equations are obtained from the anti-plane problems of the elasticity theory for a composite (piece-wise homogeneous) orthotropic (in particular, isotropic) plane slackened by crack when it reaches or intersects the dividing boundary at the right angle. Algorithms of an approximate solution are designed by the collocation method, namely the method of discrete singularities. In both cases, (when the crack reaches or crosses the dividing border) behaviour of the solutions is studied and the stress intensity coefficients at the ends of the crack are calculated. Results of numerical computations are demonstrated. According to the obtained results, hypothetical predictions of the propagation of crack are made. *© 2022 Bull. Georg. Natl. Acad. Sci.*

Singular integral equations, crack, anti-plane problem, method integral equation, collocation method, method of discrete singularities, numerical realization

In the paper, issues of approximate solution to one practical problem of the theory of elasticity are studied. We consider anti-plane problems for isotropic composite (piece-wise homogeneous) plane slackened by crack when it reaches or crosses the dividing boundary at the right angle. The first problem, when the crack reaches the dividing edge, is reduced to the singular integral equation containing a fixed-singularity for the unknown characteristic function of the crack tip opening displacement. The second problem, when the crack intersects the dividing edge, is reduced to pair of singular integral equations with a fixed-singularity concerning the unknown characteristic functions of the crack tip opening displacement. Issues of the behaviour of the solution are theoretically studied for both cases, at the endpoints of the crack and the dividing edge of materials [1,2]. The behaviour of the solutions closeness of the endpoints of the crack is studied by the method of discrete singularities with equal division of both intervals. Corresponding algorithms are designed and implemented. In both cases (when the crack reaches or crosses the dividing edge) the stress intensity coefficients are computed. The numerical results are presented and hypothetical predictions of crack propagation are made.

Statement of the Problem

Let elastic Ω body occupy complex variable plane $z = x + iy$ which is cut on the line $L = [-1,1]$. The plane consists of two orthotropic (in particular, isotropic) homogeneous half-planes $\Omega_1 = \{z \,|\, \mathrm{Re}(z) \ge 0, x \notin L_1 = [0,1]\}$ and $\Omega_2 = \{z \,|\, \mathrm{Re}(z) \le 0, x \notin L_2 = [-1,0]\}$ which are welded on the axis *y*. The following boundary value problem is considered:

$$\frac{\partial^2 w_k(x,y)}{\partial x^2} + \lambda_k^2 \frac{\partial^2 w_k(x,y)}{\partial y^2} = 0,\ (x,y) \in \Omega_k,\ k = 1,2. \tag{1}$$

On the boundary of the tangential stresses of crack it is given:

$$b_{44}^{(k)} \frac{\partial w_k(x,\pm 0)}{\partial y} = q_k^{(\pm)}(x),\ x \in L_k, \tag{2}$$

on the axis y the conditions of continuity are fulfilled:

$$w_1(0,y) = w_2(0,y),\ y \in (-\infty,+\infty),\ y \ne 0, \tag{3}$$

$$b_{55}^{(1)} \frac{\partial w_1(0,y)}{\partial x} = b_{55}^{(2)} \frac{\partial w_2(0,y)}{\partial x}, \tag{4}$$

where $\lambda_k^2 = \dfrac{b_{44}^{(k)}}{b_{55}^{(k)}}$, $b_{44}^{(k)}$ and $b_{55}^{(k)}$ are elastic constants, $q_k(x) \in H$ (here *H* denotes Hölder's class functions), $k = 1,2$. Using the theory of analytical functions with boundary conditions (2)-(4), the problem (1) is reduced to the system of singular integral equations with respect to the unknown characteristic functions of crack tip opening displacement (the tangential stress jumps) $\rho_k(x)$ [1,2]

$$\begin{cases} \displaystyle\int_0^1 \left(\frac{1}{t-x} - \frac{a_1}{t+x}\right)\rho_1(t)\,dt + b_1\int_{-1}^0 \frac{\rho_2(t)}{t-x}\,dt = 2\pi f_1(x),\ x \in [0,1], \\ \displaystyle b_2\int_0^1 \frac{\rho_1(t)}{t-x}\,dt + \int_{-1}^0 \left(\frac{1}{t-x} - \frac{a_2}{t+x}\right)\rho_2(t)\,dt = 2\pi f_2(x),\ x \in [-1,0], \end{cases} \tag{5}$$

where $\rho_k(x)$, $f_k(x)$ are unknown and given real functions, respectively. a_k and b_k are constants with

$$a_k = \frac{1-\gamma_k}{1+\gamma_k},\ b_k = \frac{2}{1+\gamma_k},\ \gamma_1 = \frac{1}{\gamma_2},\ \gamma_2 = \frac{b_{55}^{(2)}}{b_{55}^{(1)}},\ f_k(x) = \frac{\lambda_k}{b_{44}^{(k)}} q_k(x),$$

$$f_k(x) \in H,\ \rho_k(x) \in H^* \text{ (cf. [3])},\ k = 1,2.$$

The study of behavior of solutions near the ends of the crack edge has special importance. The solutions of the system of the integral equations (5) should be represented in the following way:

$$\rho_1(t) = \frac{\chi_1(t)}{t^{\alpha_1}(1-t)^{\beta_1}},\ \rho_2(t) = \frac{\chi_2(t)}{t^{\alpha_2}(1+t)^{\beta_2}} \tag{6}$$

where a_k and b_k are unknown constants, besides $\chi_k(t)$ is Hölder's class function, $k=1,2$. At the points of $t=\pm 1$, we have $\beta_1=\beta_2=\frac{1}{2}$. In the above-mentioned case, there is no singularity at the point of $t=0$ [1,2].

When one half-plane has a finite length of the linear crack, which is perpendicular to the boundary, and one end of which is located on the boundary, one singular integral equation containing a fixed-singularity is obtained in this particular case [1,2].

If $\rho_2(x)\equiv 0$, $\rho_1(x)\neq 0$, then from the system (5), we obtain one integral equation:

$$\int_0^1\left(\frac{1}{t-x}-\frac{a_1}{t+x}\right)\rho_1(t)\,dt=2\pi f_1(x),\ \ x\in[0,1]. \tag{7}$$

In such a case, we get an order of singularity at the point of $t=0$ which is dependent on elastic constants of materials and belongs to the interval $(0,1)$: $\alpha_1=1-\frac{1}{\pi}\arccos\left(\frac{b_{55}^{(1)}-b_{55}^{(2)}}{b_{55}^{(1)}+b_{55}^{(2)}}\right)\in(0,1)$, $\beta_1=\frac{1}{2}$. If $b_{55}^{(1)}=b_{55}^{(2)}$, then $\alpha_1=\frac{1}{2}$ [1,2].

As at both sides of the body, we have a solution that limits are infinities at the neighbourhood of the crack, so for the existence of a unique solution of the integral equation (7), it is necessary to add the following condition

$$\int_0^1 \rho_1(t)\,dt=C, \tag{8}$$

where $C=const$ [3,4].

Remark. The result obtained for the system of integral equations (5) by initial theoretical investigations (due to the anti-plane nature of the problem) that we have no singularity at the point of $t=0$ is wrong. As the numerical experiments demonstrate, we have a weak singularity at the point of $t=0$. In this paper, the results of numerical computations are illustrated for one particular problem. In the case of the integral equation (7), the numerical computations are also matched to the theoretical results.

Algorithm for Approximate Solution of Singular Integral Equation

Let us consider the singular integral equation (7) containing a fixed-singularity which is solved by a collocation method, in particular, a method of discrete singularities (cf. [4]). For the intervals $t\in[0,1]$ and $x\in[0,1]$, the following uniform meshes are introduced, namely, we have the following grid points: $t_{1i}=ih$, $i=1,2,\dots,n$; $x_{1j}=t_{1j}+\frac{h}{2}$, $j=1,2,\dots,n$; $h=\frac{1}{n+1}$. Using quadrature formulae [4], the equations (7) and (8) are reduced to the following one:

$$\begin{cases} h\sum_{i=1}^{n}\left(\frac{1}{t_{1i}-x_{1j}}-\frac{a_1}{t_{1i}+x_{1j}}\right)\rho_1(t_{1i})=2\pi f_1(x_{1j}),\ \ j=1,2,\dots,n-1, \\ h\sum_{i=1}^{n}\rho_1(t_{1i})=C. \end{cases} \tag{9}$$

It should be noted that we get the system (9) of *n* linear equations with *n* unknowns.

Numerical Realization of a Singular Integral Equation

In the already–mentioned problems our main goals are an investigation of a possible propagation of cracks along a body, a study behaviour of solution (the character function of stress) and finding the values of stress intensity factors (SIF) in the closeness of the endpoints of the crack. For this purpose, we have calculated values of the stress intensity factors $SIF_1=\lim_{x\to 0^+} x^{\alpha_1}\rho_1(x)\approx x_{11}^{\alpha_1}\rho_1(x_{11})$, $SIF_2=\lim_{x\to 1^-}\sqrt{1-x}\rho_1(x)\approx\sqrt{1-x_{1n}}\rho_1(x_{1n})$. Approximate values at the ends of the crack by using the algorithm with uniform division of the closed interval $[0,1]$ for which the number of interval divisions is doubled on each step. The approximate values of coefficients of the stress intensity factor in the closeness of the endpoints of the crack are computed. Consider a body composed of two isotropic materials (copper and aluminium, [5]) for the following cases:

Problem 1. Aluminium ($b_{44}^{(2)}=b_{55}^{(2)}\approx 25.00$ GPa, [6]) – Copper ($b_{44}^{(1)}=b_{55}^{(1)}\approx 46.00$ GPa, [6]). In this case copper has a crack. $\alpha_1=1-\frac{1}{\pi}\arccos\left(\frac{21}{71}\right)\approx 0.5956$. For the numerical computations of the functions $q_1(x)$ the following two cases are considered:

1. $q_1(x)\equiv 0.01$ GPa, $C=0$,
2. $q_1(x)\equiv 0.02$ GPa, $C=0$.

The numerical results for these cases are given in Table 1 and Table 2, respectively.

Table 1. The numerical values of SIF with $q=0.01$ GPa, $C=0$

n	3	7	15	31	63	127
SIF_1	-5.21×10^{-5}	-5.97×10^{-5}	-6.29×10^{-5}	-6.43×10^{-5}	-6.46×10^{-5}	-6.53×10^{-5}
SIF_2	3.38×10^{-5}	3.92×10^{-5}	4.25×10^{-5}	4.38×10^{-5}	4.49×10^{-5}	4.55×10^{-5}

Table 2. The numerical values of SIF with $q=0.02$ GPa, $C=0$

n	3	7	15	31	63	127
SIF_1	-1.04×10^{-4}	-1.19×10^{-4}	-1.26×10^{-4}	-1.29×10^{-4}	-1.29×10^{-4}	-1.31×10^{-4}
SIF_2	6.76×10^{-5}	7.84×10^{-5}	8.50×10^{-5}	8.76×10^{-5}	8.98×10^{-5}	9.10×10^{-5}

Problem 2. Copper ($b_{44}^{(2)}=b_{55}^{(2)}\approx 46.00$ GPa, [6]) – Aluminium ($b_{44}^{(1)}=b_{55}^{(1)}\approx 25.00$ GPa, [6]). Here aluminium has a crack. $\alpha_1=\frac{1}{\pi}\arccos\left(\frac{21}{71}\right)\approx 0.4044$. Consider the same cases for $q_1(x)$ and C which are considered in Problem 1.

Table 3. The numerical values of SIF with $q=0.01$ GPa, $C=0$

n	3	7	15	31	63	127
SIF_1	-1.10×10^{-4}	-1.31×10^{-4}	-1.44×10^{-4}	-1.51×10^{-4}	-1.55×10^{-4}	-1.58×10^{-4}
SIF_2	5.70×10^{-5}	6.60×10^{-5}	7.10×10^{-5}	7.30×10^{-5}	7.44×10^{-5}	7.50×10^{-5}

Table 4. The numerical values of SIF with $q = 0.02$ **GPa,** $C = 0$

n	3	7	15	31	63	127
SIF_1	-2.19×10^{-4}	-2.62×10^{-4}	-2.88×10^{-4}	-3.02×10^{-4}	-3.11×10^{-4}	-3.16×10^{-4}
SIF_2	1.14×10^{-4}	1.32×10^{-4}	1.42×10^{-4}	1.46×10^{-4}	1.49×10^{-4}	1.50×10^{-4}

Remark: In the considered problems when the absolute value of the physical content C is very small $(C \approx 0)$ then the numerical results are correspondence with a mechanical view. As we consider linear problems of the elasticity theory so increment or reduction of loads causes proportional increment or reduction of values of corresponding solutions. The last fact gives us the possibility to make hypothetical predictions about the propagations of a crack.

Algorithm on Approximate Solution of a System of the Singular Integral Equations

Let us consider the system of the singular integral equations (5) containing a fixed-singularity which is solved by a method of discrete singularities (cf. [4]). To do so, a uniform division of the closed interval $[-1,1]$ is applied. We look for solutions to the system (5) with the forms (6) [2,7]. Knots for the variables t and x are introduced as: $t_{1i} = ih$, $t_{2i} = -1 + ih$, $i = 1,2,\dots,n$; $x_{1j} = t_{1j} - \frac{h}{2}$, $x_{2j} = t_{2j} + \frac{h}{2}$, $j = 1,2,\dots,n$; $h = \frac{1}{n+1}$. The system (5) is represented by using quadrature formulae as follows (*cf.* [4]) for which $j = 1,2,\dots,n$

$$\begin{cases} h\sum_{i=1}^{n}\left(\frac{1}{t_{1i}-x_{1j}} - \frac{a_1}{t_{1i}+x_{1j}}\right)\rho_1(t_{1i}) + hb_1\sum_{i=1}^{n}\left(\frac{1}{t_{2i}-x_{1j}}\right)\rho_2(t_{2i}) = 2\pi f_1(x_{1j}), \\ hb_2\sum_{i=1}^{n}\left(\frac{1}{t_{1i}-x_{2j}}\right)\rho_1(t_{1i}) + h\sum_{i=1}^{n}\left(\frac{1}{t_{2i}-x_{2j}} - \frac{a_2}{t_{2i}+x_{2j}}\right)\rho_2(t_{2i}) = 2\pi f_2(x_{2j}). \end{cases} \tag{10}$$

Consider the following cases:

1. $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = \frac{1}{2}$ (it should be mentioned that a singularity is not at the point of $t = 0$).
2. $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\beta_1 = \beta_2 = \frac{1}{2}$ (we have a singularity at the point of $t = 0$).

Introduce the following two equations:

$$h\sum_{i=1}^{n}\rho_1(t_{1i}) = c_1, \tag{11}$$

$$h\sum_{i=1}^{n}\rho_2(t_{2i}) = c_2, \tag{12}$$

here c_1 and c_2 are constants (in our case, we assume that $c_1 = c_2 = 0$).

It should be mentioned that, in the second case, when we have a singularity at the point of $t = 0$, we replace the first and last equations in the system (10) at the knots x_{11} and x_{2n} with the equations (11) and (12), respectively. It is straightforward to deduce that the system involves 2n equations with 2n variables.

Numerical Realization of the System of Singular Integral Equations

The algorithm is approved by tests and the results of numerical computations are represented in tables. The main objectives are the same as the previous section. We have calculated approximate values of the coefficients of stress intensity factors $SIF_1 = \lim_{x\to -1^+} \sqrt{1+x}\rho_2(x) \approx \sqrt{1+x_{21}}\rho_2(x_{21})$, $SIF_2 = \lim_{x\to 1^-} \sqrt{1-x}\rho_1(x) \approx \sqrt{1-x_{1n}}\rho_1(x_{1n})$ at the ends of the crack by using the uniform division of the closed interval $[-1,1]$ for which the number of interval divisions is doubled on each step. The approximate values of coefficients of the stress intensity factors in the neighbourhood of the ends of a crack are computed. Consider a body composed of two isotropic materials copper Ω_1 and aluminium Ω_2 with elastic constants $b_{44}^{(2)} = b_{55}^{(2)} \approx 25.00$ GPa and $b_{44}^{(1)} = b_{55}^{(1)} \approx 46.00$ GPa [6], respectively. We first consider the case when $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = \dfrac{1}{2}$. Numerical computations for the functions $q_1(x)$ and $q_1(x)$ are carried out. We study the following three cases:

1. $q_1(x) = q_2(x) \equiv 0.01$ GPa,
2. $q_1(x) \equiv 0.01$ GPa, $q_2(x) \equiv 0.02$ GPa,
3. $q_1(x) \equiv 0.02$ GPa, $q_2(x) \equiv 0.01$ GPa.

The original system of singular integral equations (10) is solved by a method of discrete singularities in the interval $[-1,1]$. The results of the numerical computations of the stress intensity factors (SIF_1 and SIF_2) are given.

Table 5. The numerical values of *SIF* for two materials

Case		n			$q(x)$	Material
		15	31	63		
1	SIF_1	-3.1560×10^{-4}	-3.4338×10^{-4}	-3.6936×10^{-4}	0.01	Aluminium
	SIF_2	3.0670×10^{-5}	1.8630×10^{-5}	5.8900×10^{-6}	0.01	Copper
2	SIF_1	-7.9752×10^{-4}	-8.8447×10^{-4}	-9.6753×10^{-4}	0.02	Aluminium
	SIF_2	-1.3563×10^{-4}	-1.7907×10^{-4}	-2.2287×10^{-4}	0.01	Copper
3	SIF_1	-1.4929×10^{-4}	-1.4567×10^{-4}	-1.4060×10^{-4}	0.01	Aluminium
	SIF_2	2.2765×10^{-4}	2.3498×10^{-4}	2.4056×10^{-4}	0.02	Copper

Remark: Due to the anti-plane nature of the problem we assume that we do not have a singularity at the point of $t = 0$. Numerical results using the method of discrete singularities have given improper results (see Table 5). Because of this reason we have introduced some assumptions. If we have a singularity at the point of $t = 0$, we solve the system (10) taking the equations (11) and (12) into account.

In the second case, it is considered the following assumptions $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\beta_1 = \beta_2 = \dfrac{1}{2}$.

Table 6. The numerical values of *SIF* with $q_1 = q_2 = 0.01$

n	3	7	15	31	63	127
SIF_1	-1.04×10^{-4}	-1.23×10^{-4}	-1.35×10^{-4}	-1.42×10^{-4}	-1.47×10^{-4}	-1.50×10^{-4}
SIF_2	6.18×10^{-5}	7.52×10^{-5}	8.39×10^{-5}	8.97×10^{-5}	9.38×10^{-5}	9.68×10^{-5}

Table 7. The numerical values of SIF with $q_1 = q_2 = 0.02$

n	3	7	15	31	63	127
SIF_1	-2.07×10^{-4}	-2.46×10^{-4}	-2.69×10^{-4}	-2.84×10^{-4}	-2.93×10^{-4}	-3.00×10^{-4}
SIF_2	1.24×10^{-4}	1.50×10^{-4}	1.68×10^{-4}	1.80×10^{-4}	1.88×10^{-4}	1.94×10^{-4}

In the considered assumptions, we obtain relevant results which means that in the neighbourhood of endpoints of a crack the absolute values of stress intensity factors must be increased. Indeed, Tables 6 and 7 demonstrate this reality.

Conclusion

Equations of the anti-plane elasticity theory for composite bodies slackened by crack should be used as an initial approximation of the mathematical model. More interesting cases when a crack intersects an interface or intersects the boundary at a rectangular angle are investigated. As the system of the integral equations, as well as an integral equation, are solved by a method of discrete singularities. An algorithm for finding approximate solutions for the boundary value problems for the singular integral equations containing a fixed-singularity is worked out. The system of linear algebraic equations is solved using computer software. For investigation of the rate of cracks propagation, we have calculated the stress intensity factors for different materials. If the absolute values of the stress intensity factors are the neighbourhood of 1 they are critical limits of the spreading of cracks. If the absolute values of the stress intensity factors are essentially less than 1, then cracks are not almost developed [8]. According to the numerical experiments of considered linear problems of elasticity theory, a hypothetical prediction should be made on the propagation of a crack.

მათემატიკა

დრეკადობის თეორიის ანტიბრტყელი ამოცანების რიცხვითი ამოხსნა წრფივი ბზარით შესუსტებული კომპოზიტური იზოტროპული სიბრტყისთვის

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წინამდებარე ნაშრომში შესწავლილია სინგულარული ინტეგრალური განტოლების მიახლოებითი ამონახსნის საკითხები და უძრავი სინგულარობის შემცველი ინტეგრალურ განტოლებათა სისტემები. შესწავლილი ინტეგრალური განტოლებები მიღებულია დრეკადობის თეორიის ანტიბრტყელი ამოცანებიდან კომპოზიტური (ნაწილობრივად ერთგვაროვანი) ორთოტროპული (კერძოდ, იზოტროპული) სიბრტყისთვის, რომელიც შესუსტებულია ბზარით, როდესაც ის აღწევს ან კვეთს გამყოფ საზღვარს მართი კუთხით. მიახლოებითი ამოხსნის ალგორითმები შემუშავებულია კოლოკაციის მეთოდის გამოყენებით, კერძოდ, დისკრეტულ განსაკუთრებულობათა მეთოდით. ორივე შემთხვევაში (როდესაც ბზარი აღწევს ან კვეთს გამყოფ საზღვარს) შესწავლილია ამონახსნების ყოფაქცევა და გამოთვლილია დაძაბულობის ინტენსივობის ფაქტორები ბზარის ბოლოებზე. წარმოდგენილია რიცხვითი გამოთვლების შედეგები. მიღებული შედეგების მიხედვით გაკეთებულია ბზარის გავრცელების ჰიპოთეტური პროგნოზები.

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