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Mathematics

On Approximation of Nonclassical Model for Thermoelastic Bars by One-Dimensional Problems

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In the present paper, variational formulation of the three-dimensional initial boundary value problem corresponding to the Green-Lindsay nonclassical model for thermoelastic bars with variable cross-section is considered. An algorithm of approximation of the nonclassical dynamic three-dimensional model for bar by a sequence of one-dimensional problems is constructed, when density of surface forces and heat flux along the outward normal vector of the boundary are given on the lateral face surfaces of the bar. The constructed one-dimensional initial-boundary value problems are investigated in suitable function spaces, the convergence in corresponding spaces of the sequence of vector-functions of three space variables, restored from the solutions of one-dimensional problems, to the solution of the original three-dimensional problem is proved and under additional conditions the error of approximation is estimated. *© 2022 Bull. Georg. Natl. Acad. Sci.*

nonclassical thermoelasticity, bar, double Fourier-Legendre series, approximation algorithm

One of the nonclassical models for thermoelastic solids devoted to elimination the shortcomings of classical thermoelasticity was obtained by A. Green and K. Lindsay [1], which is described by a system of partial differential equations of second order with respect to the time variable and depend on two relaxation times. By applying the method of potential and theory of integral equations the problems of stable and pseudo oscillations for the Green-Lindsay nonclassical model were studied in [2]. The problem of propagation of a thermoelastic wave was studied and a domain of influence result was obtained in [3] in classical spaces of twice continuously differentiable functions. The initial-boundary value problem with mixed boundary conditions corresponding to the Green–Lindsay linear dynamic three-dimensional model for general inhomogeneous anisotropic thermoelastic body was investigated in Sobolev spaces in [4].

Since boundary and initial-boundary value problems defined on two-dimensional and one-dimensional space domains are frequently used instead of three-dimensional problems, it is important to construct algorithms of approximation of the original problems by lower-dimensional ones. One of dimensional reduction methods for plates with variable thickness in the classical theory of elasticity was suggested by I. Vekua in the paper [5]. Mathematical results on the relationship between the two-dimensional hierarchical models constructed in [5] and the three-dimensional ones for static and dynamic problems in Sobolev spaces were obtained in [6]. Two-dimensional hierarchical models for thermoelastic plates in the framework of the Green-Lindsay model were constructed and investigated in [7]. Various hierarchical models were constructed and investigated applying Vekua's reduction method and its generalizations (see [8-14] and references given therein).

This paper is devoted to construction and investigation of an algorithm of approximation of Green-Lindsay nonclassical three-dimensional model for thermoelastic bar with variable rectangular cross-section, which may vanish on the butt end, by one-dimensional problems. We consider the variational formulation of the three-dimensional initial-boundary value problem for thermoelastic bar and construct a hierarchy of one-dimensional problems in Sobolev spaces, when temperature and displacement vanishes along the butt end of the bar with positive area and density of surface forces and heat flux along the outward normal vector are given on the remaining part of the boundary. We investigate the existence and uniqueness of solutions of the constructed one-dimensional problems in suitable spaces of vector-valued distributions with values in weighted Sobolev spaces. Moreover, we prove that the sequence of vector-functions of three space variables restored from the solutions of the one-dimensional problems converges in corresponding spaces to the solution of the original three-dimensional problem and, if it possesses additional regularity, we estimate the rate of convergence.

We denote by $W^{r,2}(D) = H^r(D)$ and $H^r(\hat{\Gamma})$, $r \in \mathbf{R}$, the Sobolev spaces of order r based on the spaces $H^0(D) = L^2(D)$ and $H^0(\hat{\Gamma}) = L^2(\hat{\Gamma})$ of square-integrable functions, respectively, where $D \subset \mathbf{R}^p$, $p \in \mathbf{N}$, is a bounded Lipschitz domain [15] and $\hat{\Gamma} \subset \partial D$ is a Lipschitz surface. We denote by $\mathbf{H}^r(D) = [H^r(D)]^3$, $\mathbf{H}^r(\hat{\Gamma}) = [H^r(\hat{\Gamma})]^3$, $\mathbf{L}^2(D) = [L^2(D)]^3$, $\mathbf{L}^s(\hat{\Gamma}) = [L^s(\hat{\Gamma})]^3$, $s \geq 1$, $r, s \in \mathbf{R}$, the corresponding spaces of vector-valued functions. The trace operators are denoted by $tr_{\hat{\Gamma}} : H^1(D) \to H^{1/2}(\hat{\Gamma})$ and $\mathbf{tr}_{\hat{\Gamma}} : \mathbf{H}^1(D) \to \mathbf{H}^{1/2}(\hat{\Gamma})$. For a Banach space X, we denote by $C^0([0,T]; X)$ the space of continuous functions on $[0,T]$ with values in X. $L^q(0,T;X)$, $1 \leq q \leq \infty$, is the space of such measurable functions $g:(0,T) \to X$ that $\|g(t)\|_X \in L^q(0,T)$ and the generalized derivative of g is denoted by $g' = dg/dt$ [16].

Let us consider a thermoelastic bar $\overline{\Omega}$ with variable rectangular cross-section with thickness or width that may vanish on the lower butt end, i.e. the initial configuration $\overline{\Omega}$ of the bar is a closure of Lipschitz domain of the following form

$$\Omega = \left\{ x = (x_1, x_2, x_3) \in \mathbf{R}^3;\, h_1^-(x_3) < x_1 < h_1^+(x_3),\, h_2^-(x_3) < x_2 < h_2^+(x_3), \quad x_3 \in I = (h_3^-, h_3^+) \right\},$$

where $h_1^{\pm}, h_2^{\pm} \in C^0(\overline{I}) \cap C^{1,1}_{loc}((h_3^-, h_3^+])$ are continuous on $\overline{I}$ and Lipschitz continuous on $(h_3^-, h_3^+]$ together with the first order derivatives, $h_1^+(x_3) > h_1^-(x_3)$ and $h_2^+(x_3) > h_2^-(x_3)$, for $x_3 \in (h_3^-, h_3^+]$, $h_1^{\pm}(x_3)$ and $h_2^{\pm}(x_3)$ are equal or different for $x_3 = h_3^-$. The upper butt end of the bar $\overline{\Omega}$, defined by the equation $x_3 = h_3^+$ we denote by $\Gamma_0 = \{x \in \mathbf{R}^3; h_\alpha^-(x_3) \leq x_\alpha \leq h_\alpha^+(x_3), \alpha = 1,2, x_3 = h_3^+\}$ and the remaining part of the boundary $\Gamma = \partial\Omega$ we denote by $\Gamma_1 = \Gamma \setminus \Gamma_0$.

We assume that bar consists of homogeneous isotropic thermoelastic material. The applied body force density we denote by $\mathbf{f}=(f_i)_{i=1}^3:\Omega\times(0,T)\to\mathbf{R}^3$, and the density of heat sources we denote by $f^\theta:\Omega\times(0,T)\to\mathbf{R}$. The bar is clamped and temperature θ vanishes along the upper butt end Γ_0 and, on the remaining part Γ_1 of the boundary, the surface forces with density $\mathbf{g}=(g_i)_{i=1}^3:\Gamma_1\times(0,T)\to\mathbf{R}^3$ and the heat flux along the outward normal vector of the boundary with density $g^\theta:\Gamma_1\times(0,T)\to\mathbf{R}$ are given.

The nonclassical dynamic linear three-dimensional model of stress-strain state of thermoelastic body obtained by A. Green and K. Lindsay in differential form is given by the following initial-boundary value problem [1, 4]:

$$\rho\frac{\partial^2 u_i}{\partial t^2}-\sum_{j=1}^3\frac{\partial}{\partial x_j}\left(\lambda\sum_{p=1}^3 e_{pp}(\mathbf{u})\delta_{ij}+2\mu e_{ij}(\mathbf{u})+\eta\theta\delta_{ij}+\eta\tau_1\frac{\partial\theta}{\partial t}\delta_{ij}\right)=f_i \quad \text{in } \Omega\times(0,T),\ i=1,2,3, \tag{1}$$

$$\chi\left(\frac{\partial\theta}{\partial t}+\tau_0\frac{\partial^2\theta}{\partial t^2}\right)-2\beta\sum_{p=1}^3\frac{\partial^2\theta}{\partial t\partial x_p}-\kappa\sum_{j=1}^3\frac{\partial^2\theta}{\partial x_j^2}-\Theta_0\eta\frac{\partial}{\partial t}\sum_{p=1}^3 e_{pp}(\mathbf{u})=f^\theta \quad \text{in } \Omega\times(0,T), \tag{2}$$

$$\mathbf{u}=\mathbf{0} \quad \text{on } \Gamma_0\times(0,T),\quad \sum_{j=1}^3\left(\lambda\sum_{p=1}^3 e_{pp}(\mathbf{u})\delta_{ij}+2\mu e_{ij}(\mathbf{u})+\eta\theta\delta_{ij}+\eta\tau_1\frac{\partial\theta}{\partial t}\delta_{ij}\right)n_j=g_i \quad \text{on } \Gamma_1\times(0,T), \tag{3}$$

$$\theta=0 \quad \text{on } \Gamma_0\times(0,T),\quad -\sum_{j=1}^3\left(\kappa\frac{\partial\theta}{\partial x_j}+\beta\frac{\partial\theta}{\partial t}\right)n_j=g^\theta \quad \text{on } \Gamma_1\times(0,T), \tag{4}$$

$$u_i(x,0)=u_{0i}(x),\ \frac{\partial u_i}{\partial t}(x,0)=u_{1i}(x),\ \theta(x,0)=\theta_0(x),\ \frac{\partial\theta}{\partial t}(x,0)=\theta_1(x) \quad \text{in } \Omega,\quad i=1,2,3, \tag{5}$$

where $\mathbf{n}=(n_i)_{i=1}^3$ is the unit outward normal vector to Γ, δ_{ij} is the Kronecker's delta, $\mathbf{u}=(u_i)_{i=1}^3:\Omega\times(0,T)\to\mathbf{R}^3$ is the displacement vector-function of thermoelastic body, $\theta:\Omega\times(0,T)\to\mathbf{R}$ is the temperature distribution, λ,μ are Lamé constants, $\rho>0$ is a mass density, $\kappa>0$ is a heat conductivity coefficient, $\chi>0$ is a volumetric heat, η is a thermoelastic constant, β is a thermal coefficient, $\Theta_0>0$ is a temperature of the medium in the natural state of no deformation, which is considered as a reference temperature, τ_0,τ_1 are relaxation times, and $\mathbf{u}_0=(u_{0i})_{i=1}^3$ and $\mathbf{u}_1=(u_{1i})_{i=1}^3$ are the initial displacement and velocity vector-functions, θ_0 and θ_1 are the initial distribution of temperature and rate of change of temperature, $e_{ij}(\mathbf{v})=1/2(\partial v_i/\partial x_j+\partial v_j/\partial x_i)$, $i,j=1,2,3$, $\mathbf{v}=(v_i)_{i=1}^3$, is the strain tensor. Note, that in the case of $\tau_0=\tau_1=0$ the nonclassical three-dimensional model (1)-(5) coincides with the classical linear three-dimensional model for thermoelastic bodies.

In order to construct a sequence of one-dimensional problems approximating the dynamical three-dimensional model let us consider the following variational formulation in the spaces of vector-valued distributions, which is equivalent to the problem (1)-(5) in the spaces of smooth enough functions: Find $\mathbf{u}\in C^0([0,T];\mathbf{V}(\Omega))$, $\mathbf{u}'\in L^\infty(0,T;\mathbf{V}(\Omega))$, $\mathbf{u}''\in L^\infty(0,T;\mathbf{L}^2(\Omega))$, $\theta\in C^0([0,T];V^\theta(\Omega))$, $\theta'\in L^\infty(0,T;V^\theta(\Omega))$, $\theta''\in L^\infty(0,T;L^2(\Omega))$, which satisfies the following equations in the sense of distributions on $(0,T)$,

$$(\rho\mathbf{u}''(.),\mathbf{v})_{\mathbf{L}^2(\Omega)}+a(\mathbf{u}(.),\mathbf{v})+b(\theta(.)+\tau_1\theta'(.),\mathbf{v})=(\mathbf{f}(.),\mathbf{v})_{\mathbf{L}^2(\Omega)}+(\mathbf{g}(.),\mathbf{tr}_{\Gamma_1}(\mathbf{v}))_{\mathbf{L}^2(\Gamma_1)},\quad \forall\mathbf{v}\in\mathbf{V}(\Omega), \tag{6}$$

$$\begin{aligned}&(\chi\tau_0\theta''(.),\varphi)_{L^2(\Omega)}+(\chi\theta'(.),\varphi)_{L^2(\Omega)}-b^\theta(\theta'(.),\varphi)+a^\theta(\theta(.),\varphi)+\\&+b^\theta(\varphi,\theta'(.))-\Theta_0 b(\varphi,\mathbf{u}'(.))=(f^\theta(.),\varphi)_{L^2(\Omega)}-(g^\theta(.),tr_{\Gamma_1}(\varphi))_{L^2(\Gamma_1)},\quad \forall\varphi\in V^\theta(\Omega),\end{aligned} \tag{7}$$

together with the initial conditions

$$\mathbf{u}(0)=\mathbf{u}_0, \qquad \mathbf{u}'(0)=\mathbf{u}_1, \qquad \theta(0)=\theta_0, \qquad \theta'(0)=\theta_1, \tag{8}$$

where $\mathbf{V}(\Omega)=\{\mathbf{v}\in\mathbf{H}^1(\Omega);\mathbf{tr}(\mathbf{v})=\mathbf{0}\ \text{ on }\Gamma_0\}$, $V^\theta(\Omega)=\{\varphi\in H^1(\Omega);tr(\varphi)=0\ \text{ on }\Gamma_0\}$,

$$a(\tilde{\mathbf{v}},\mathbf{v})=\int_\Omega\left(\lambda\sum_{p=1}^3 e_{pp}(\tilde{\mathbf{v}})\sum_{q=1}^3 e_{qq}(\mathbf{v})+2\mu\sum_{i,j=1}^3 e_{ij}(\tilde{\mathbf{v}})e_{ij}(\mathbf{v})\right)dx, \quad \forall\mathbf{v},\tilde{\mathbf{v}}\in\mathbf{V}(\Omega),$$

$$a^\theta(\bar\varphi,\varphi)=\kappa\int_\Omega\sum_{j=1}^3\frac{\partial\bar\varphi}{\partial x_j}\frac{\partial\varphi}{\partial x_j}dx, \quad \forall\varphi,\bar\varphi\in V^\theta(\Omega),$$

$$b^\theta(\varphi,\tilde\varphi)=\beta\left(\sum_{p=1}^3\frac{\partial\varphi}{\partial x_p},\tilde\varphi\right)_{L^2(\Omega)}, \; b(\tilde\varphi,\mathbf{v})=\eta\left(\tilde\varphi,\sum_{p=1}^3\frac{\partial v_p}{\partial x_p}\right)_{L^2(\Omega)}, \quad \forall\varphi\in H^1(\Omega),\tilde\varphi\in L^2(\Omega),\mathbf{v}\in\mathbf{H}^1(\Omega),$$

$(.,.)_{\mathbf{L}^2(\Omega)}$, $(.,.)_{L^2(\Omega)}$, $(.,.)_{\mathbf{L}^2(\Gamma_1)}$ and $(.,.)_{L^2(\Gamma_1)}$ are scalar products in the spaces $\mathbf{L}^2(\Omega)$, $L^2(\Omega)$, $\mathbf{L}^2(\Gamma_1)$ and $L^2(\Gamma_1)$, respectively.

For the formulated initial-boundary value problem (6)-(8), corresponding to the Green-Lindsay dynamic three-dimensional model, the following theorem is valid [4].

Theorem 1. Suppose that Ω is a bounded Lipschitz domain and $\mu>0$, $3\lambda+2\mu>0$, $\rho>0$, $\kappa>0$, $\chi>0$, $\tau_0>0$, $\tau_1>0$. If $\mathbf{f},\mathbf{f}'\in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g},\mathbf{g}',\mathbf{g}''\in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, $f^\theta, f^{\theta'}\in L^2(0,T;L^2(\Omega))$, $g^\theta, g^{\theta'}, g^{\theta''}\in L^2(0,T;L^{4/3}(\Gamma_1))$ and initial conditions $\mathbf{u}_0\in\mathbf{H}^2(\Omega)\cap\mathbf{V}(\Omega)$, $\mathbf{u}_1\in\mathbf{V}(\Omega)$, $\theta_0\in H^2(\Omega)\cap V^\theta(\Omega)$, $\theta_1\in V^\theta(\Omega)$ satisfy the following compatibility conditions

$$\begin{aligned} g_i(0)&=\sum_{j=1}^3 tr_{\Gamma_1}\left(\lambda\sum_{p=1}^3 e_{pp}(\mathbf{u}_0)\delta_{ij}+2\mu e_{ij}(\mathbf{u}_0)+\eta\theta_0\delta_{ij}+\eta\tau_1\theta_1\delta_{ij}\right)n_j, \; i=1,2,3,\\ g^\theta(0)&=-\sum_{j=1}^3 tr_{\Gamma_1}\left(\kappa\frac{\partial\theta_0}{\partial x_j}+\beta\theta_1\right)n_j, \end{aligned} \tag{9}$$

then the initial-boundary value problem (6)-(8) possesses a unique solution.

In order to construct an algorithm of approximation of the Green-Lindsay nonclassical three-dimensional model by one-dimensional problems, let us consider the subspaces $\mathbf{V}^2_{\mathbf{N}^1\mathbf{N}^2}(\Omega)\subset\mathbf{H}^2(\Omega)\cap\mathbf{V}(\Omega)$, $\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)\subset\mathbf{V}(\Omega)$ and $\mathbf{H}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)\subset\mathbf{L}^2(\Omega)$, $\mathbf{N}^1=(N_1^1,N_2^1,N_3^1)$, $\mathbf{N}^2=(N_1^2,N_2^2,N_3^2)$, consisting of polynomials with respect to the variables x_1 and x_2,

$$\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}=(v_{\mathbf{N}^1\mathbf{N}^2 i})_{i=1}^3, \quad v_{\mathbf{N}^1\mathbf{N}^2 i}=\sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=0}^{N_i^2}\frac{1}{h_1h_2}\left(r_i^1+\frac{1}{2}\right)\left(r_i^2+\frac{1}{2}\right)\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}P_{r_i^1}(z_1)P_{r_i^2}(z_2),$$

where $(h_1h_2)^{-1/2}\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}\in L^2(I)$, $0\le r_i^\alpha\le N_i^\alpha$, $z_\alpha=\dfrac{x_\alpha-\bar h_\alpha}{h_\alpha}$, $h_\alpha=\dfrac{h_\alpha^+-h_\alpha^-}{2}$, $\bar h_\alpha=\dfrac{h_\alpha^++h_\alpha^-}{2}$, $\alpha=1,2$, $i=1,2,3$, and P_r is the Legendre polynomial of order $r\in\mathbf{N}\cup\{0\}$. We also consider the subspaces $V^{\theta,2}_{N_\theta^1N_\theta^2}(\Omega)\subset H^2(\Omega)\cap V^\theta(\Omega)$, $V^\theta_{N_\theta^1N_\theta^2}(\Omega)\subset V^\theta(\Omega)$ and $H^\theta_{N_\theta^1N_\theta^2}(\Omega)\subset L^2(\Omega)$, consisting of the functions

$$\varphi_{N_\theta^1N_\theta^2}=\sum_{r^1=0}^{N_\theta^1}\sum_{r^2=0}^{N_\theta^2}\frac{1}{h_1h_2}\left(r^1+\frac{1}{2}\right)\left(r^2+\frac{1}{2}\right)\overset{r^1r^2}{\varphi}_{N_\theta^1N_\theta^2}P_{r^1}(z_1)P_{r^2}(z_2),$$

where $(h_1h_2)^{-1/2}\overset{r^1r^2}{\varphi}_{N_\theta^1N_\theta^2}\in L^2(I),\ \ 0\le r^1\le N_\theta^1,\ \ 0\le r^2\le N_\theta^2$.

Since the functions $h_1^{\pm}$ and $h_2^{\pm}$ are Lipschitz continuous together with their first order derivatives in the interior of the interval I, from Rademacher's theorem [17] we have that $h_1^{\pm}$, $h_2^{\pm}$, $(h_1^{\pm})'$, $(h_2^{\pm})'$, are differentiable almost everywhere in I and $h_\alpha^{\pm},(h_\alpha^{\pm})',(h_\alpha^{\pm})''\in L^\infty(I^*)$, $\alpha=1,2,$ for all $I^*=(h_3^{-,*},h_3^{+,*})$, $h_3^-<h_3^{-,*}<h_3^{+,*}<h_3^+$. Therefore, since $h_1^{\pm}$ and $h_2^{\pm}$ are positive I we have that, for any vector-function $\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}=(v_{\mathbf{N}^1\mathbf{N}^2i})_{i=1}^3\in\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}^2(\Omega)$, the corresponding functions $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}\in H^2(I^*)$, for all I^*, $\overline{I^*}\subset I$, i.e. $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}\in H_{loc}^2(I)$, $0\le r_i^1\le N_i^1$, $0\le r_i^2\le N_i^2$, $i=1,2,3$. Similarly, if $\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}\in\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, then $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}\in H^1(I^*)$ for all I^*, $\overline{I^*}\subset I$, i.e. $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}\in H_{loc}^1(I)$, $0\le r_i^1\le N_i^1$, $0\le r_i^2\le N_i^2$, $i=1,2,3$. Similarly, for functions from the spaces $V_{N_\theta^1N_\theta^2}^{\theta,2}(\Omega)$ and $V_{N_\theta^1N_\theta^2}^{\theta}(\Omega)$ we also have $\overset{r^1r^2}{\varphi}_{N_\theta^1N_\theta^2}\in H_{loc}^2(I)$, if $\varphi_{N_\theta^1N_\theta^2}\in V_{N_\theta^1N_\theta^2}^{\theta,2}(\Omega)$ and $\overset{r^1r^2}{\varphi}_{N_\theta^1N_\theta^2}\in H_{loc}^1(I)$, if $\varphi_{N_\theta^1N_\theta^2}\in V_{N_\theta^1N_\theta^2}^{\theta}(\Omega)$, $0\le r^1\le N_\theta^1$, $0\le r^2\le N_\theta^2$. Moreover, the norms $\|\cdot\|_{\mathbf{H}^2(\Omega)}$, $\|\cdot\|_{\mathbf{H}^1(\Omega)}$ and $\|\cdot\|_{H^2(\Omega)}$, $\|\cdot\|_{H^1(\Omega)}$ in the spaces $\mathbf{H}^2(\Omega)$, $\mathbf{H}^1(\Omega)$ and $H^2(\Omega)$, $H^1(\Omega)$ define the weighted norms $\|\cdot\|_{**}$, $\|\cdot\|_{*}$ and $\|\cdot\|_{\theta**}$, $\|\cdot\|_{\theta*}$ of vector-functions $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}=(\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i})\in[H_{loc}^2(I)]^{N_{1,2,3}^{1,2}}$, $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\in[H_{loc}^1(I)]^{N_{1,2,3}^{1,2}}$, $N_{1,2,3}^{1,2}=\sum_{i=1}^3(N_i^1+1)(N_i^2+1)$, and $\vec{\varphi}_{N_\theta^1N_\theta^2}\in[H_{loc}^2(I)]^{N_\theta^{1,2}}$, $\vec{\varphi}_{N_\theta^1N_\theta^2}\in[H_{loc}^1(I)]^{N_\theta^{1,2}}$, $N_\theta^{1,2}=(N_\theta^1+1)(N_\theta^2+1)$ such that $\|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_{**}=\|\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}\|_{\mathbf{H}^2(\Omega)}$, $\|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_{*}=\|\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}\|_{\mathbf{H}^1(\Omega)}$, $\|\vec{\varphi}_{N_\theta^1N_\theta^2}\|_{\theta**}=\|\varphi_{N_\theta^1N_\theta^2}\|_{H^2(\Omega)}$, $\|\vec{\varphi}_{N_\theta^1N_\theta^2}\|_{\theta*}=\|\varphi_{N_\theta^1N_\theta^2}\|_{H^1(\Omega)}$. Using the properties of the Legendre polynomials [12], we can obtain explicit expressions for the norms $\|\cdot\|_{**}$, $\|\cdot\|_{*}$, $\|\cdot\|_{\theta**}$ and $\|\cdot\|_{\theta*}$, particularly, the norm $\|\cdot\|_{*}$ is given by the following expression:

$$
\begin{aligned}
\|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_*^2=\sum_{i=1}^3\sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=0}^{N_i^2}\left(r_i^1+\frac{1}{2}\right)\left(r_i^2+\frac{1}{2}\right)\Bigg[\left\|\frac{\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}}{\sqrt{h_1h_2}}\right\|_{L^2(I)}^2+\sum_{\alpha=1}^2\Bigg\|\sum_{k_i^\alpha=r_i^\alpha}^{N_i^\alpha}\left(k_i^\alpha+\frac{1}{2}\right)\frac{1-(-1)^{k_i^\alpha+r_i^\alpha}}{h_1h_2\sqrt{h_\alpha}}\times \\
\times\left((2-\alpha)\overset{k_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}+(\alpha-1)\overset{r_i^1k_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}\right)\Bigg\|_{L^2(I)}^2+\Bigg\|\frac{1}{\sqrt{h_1h_2}}\Bigg((\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i})'-\sum_{\alpha=1}^2\frac{(h_\alpha)'}{h_\alpha}(r_i^\alpha+1)\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}- \\
-\sum_{\alpha=1}^2\sum_{k_i^\alpha=r_i^\alpha+1}^{N_i^\alpha}\left(k_i^\alpha+\frac{1}{2}\right)\frac{(h_\alpha^+)'-(-1)^{k_i^\alpha-r_i^\alpha}(h_\alpha^-)'}{h_\alpha}\left((2-\alpha)\overset{k_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}+(\alpha-1)\overset{r_i^1k_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}\right)\Bigg)\Bigg\|_{L^2(I)}^2\Bigg],
\end{aligned}
$$

where we assume that the sums with the lower limit greater than the upper one equal to zero.

For components $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}$ and $\overset{r^1r^2}{\varphi}_{N_\theta^1N_\theta^2}$ of $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}$ and $\vec{\varphi}_{N_\theta^1N_\theta^2}$, which possess the properties $\|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_*<\infty$ and $\|\vec{\varphi}_{N_\theta^1N_\theta^2}\|_{\theta*}<\infty$, we can define the traces for $x_3=h_3^+$. Indeed, the corresponding vector-function of three space variables $\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}=(v_{\mathbf{N}^1\mathbf{N}^2i})_{i=1}^3$ and function $\varphi_{N_\theta^1N_\theta^2}$ belong to the spaces $\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)\subset\mathbf{H}^1(\Omega)$ and $V_{N_\theta^1N_\theta^2}^{\theta}(\Omega)\subset H^1(\Omega)$, respectively, and using the trace operator tr_{Γ_0} we define the traces of $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}$ and $\overset{r^1r^2}{\varphi}_{N_\theta^1N_\theta^2}$ for $x_3=h_3^+$, in particular,

$$tr_{h_3^+}(\overset{r_i^1 r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}) = \int_{h_2^-}^{h_2^+}\int_{h_1^-}^{h_1^+} tr_{\Gamma_0}(v_{\mathbf{N}^1\mathbf{N}^2 i}) P_{r_i^1}(z_1) P_{r_i^2}(z_2) dx_1 dx_2, \quad r_i^1 = 0,..,N_i^1,\ r_i^2 = 0,..,N_i^2,\ i = 1,2,3.$$

Since the vector-functions $\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2} = (v_{\mathbf{N}^1\mathbf{N}^2 i})_{i=1}^3$ from the subspaces $\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$ and $\mathbf{H}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, and the functions $\varphi_{N_\theta^1 N_\theta^2}$ from $V^\theta_{N_\theta^1 N_\theta^2}(\Omega)$ and $H^\theta_{N_\theta^1 N_\theta^2}(\Omega)$ are uniquely defined by functions $\overset{r_i^1 r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}$ and $\overset{r^1 r^2}{\varphi}_{N_\theta^1 N_\theta^2}$ of one space variable, on the constructed subspaces we obtain the following hierarchy of one-dimensional initial-boundary value problems: Find $\vec{w}_{\mathbf{N}^1\mathbf{N}^2} \in C^0([0,T];\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I))$, $\vec{w}'_{\mathbf{N}^1\mathbf{N}^2} \in L^\infty(0,T;\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I))$, $\vec{w}''_{\mathbf{N}^1\mathbf{N}^2} \in L^\infty(0,T;\vec{H}_{\mathbf{N}^1\mathbf{N}^2}(I))$, $\vec{\zeta}_{N_\theta^1 N_\theta^2} \in C^0([0,T];\vec{V}^\theta_{N_\theta^1 N_\theta^2}(I))$, $\vec{\zeta}'_{N_\theta^1 N_\theta^2} \in L^\infty(0,T;\vec{V}^\theta_{N_\theta^1 N_\theta^2}(I))$, $\vec{\zeta}''_{N_\theta^1 N_\theta^2} \in L^\infty(0,T;\vec{H}^\theta_{N_\theta^1 N_\theta^2}(I))$, which satisfy the following equations in the sense of distributions on $(0,T)$,

$$\begin{aligned} &R_{\mathbf{N}^{1,2}}(\vec{w}''_{\mathbf{N}^1\mathbf{N}^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) + a_{\mathbf{N}^{1,2}}(\vec{w}_{\mathbf{N}^1\mathbf{N}^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) + \\ &\qquad + b_{N_\theta^{1,2}\mathbf{N}^{1,2}}(\vec{\zeta}_{N_\theta^1 N_\theta^2} + \tau_1\vec{\zeta}'_{N_\theta^1 N_\theta^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = L_{\mathbf{N}^{1,2}}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}), \qquad \forall \vec{v}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I), \end{aligned} \tag{10}$$

$$\begin{aligned} &R^\theta_{N_\theta^{1,2}}(\tau_0\vec{\zeta}''_{N_\theta^1 N_\theta^2} + \vec{\zeta}'_{N_\theta^1 N_\theta^2}, \vec{\varphi}_{N_\theta^1 N_\theta^2}) - b^\theta_{N_\theta^{1,2}}(\vec{\zeta}'_{N_\theta^1 N_\theta^2}, \vec{\varphi}_{N_\theta^1 N_\theta^2}) + a^\theta_{N_\theta^{1,2}}(\vec{\zeta}_{N_\theta^1 N_\theta^2}, \vec{\varphi}_{N_\theta^1 N_\theta^2}) + \\ &\qquad + b^\theta_{N_\theta^{1,2}}(\vec{\varphi}_{N_\theta^1 N_\theta^2}, \vec{\zeta}'_{N_\theta^1 N_\theta^2}) - \Theta_0 b_{N_\theta^{1,2}\mathbf{N}^{1,2}}(\vec{\varphi}_{N_\theta^1 N_\theta^2}, \vec{w}'_{\mathbf{N}^1\mathbf{N}^2}) = L^\theta_{N_\theta^{1,2}}(\vec{\varphi}_{N_\theta^1 N_\theta^2}), \qquad \forall \vec{\varphi}_{N_\theta^1 N_\theta^2} \in \vec{V}^\theta_{N_\theta^1 N_\theta^2}(I), \end{aligned} \tag{11}$$

and the initial conditions

$$\vec{w}_{\mathbf{N}^1\mathbf{N}^2}(0) = \vec{w}_{\mathbf{N}^1\mathbf{N}^2 0}, \quad \vec{w}'_{\mathbf{N}^1\mathbf{N}^2}(0) = \vec{w}_{\mathbf{N}^1\mathbf{N}^2 1}, \quad \vec{\zeta}_{N_\theta^1 N_\theta^2}(0) = \vec{\zeta}_{N_\theta^1 N_\theta^2 0}, \quad \vec{\zeta}'_{N_\theta^1 N_\theta^2}(0) = \vec{\zeta}_{N_\theta^1 N_\theta^2 1}, \tag{12}$$

where $\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I) = \{\vec{v}_{\mathbf{N}^1\mathbf{N}^2} = (\overset{r_i^1 r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}) \in [H^1_{loc}(I)]^{N^{1,2}_{1,2,3}};\ \|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_* < \infty, tr_{h_3^+}(\overset{r_i^1 r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}) = 0,\ 0 \le r_i^\alpha \le N_i^\alpha,\ \alpha = 1,2,\ i = 1,2,3\}$,
$\vec{H}_{\mathbf{N}^1\mathbf{N}^2}(I) = \{\vec{v}_{\mathbf{N}^1\mathbf{N}^2} \in [L^2(I)]^{N^{1,2}_{1,2,3}}; (h_1 h_2)^{-1/2}\overset{r_i^1 r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i} \in L^2(I), r_i^1 = 0,...,N_i^1, r_i^2 = 0,...,N_i^2, i = 1,2,3\}$,
$\vec{V}^\theta_{N_\theta^1 N_\theta^2}(I) = \{\vec{\varphi}_{N_\theta^1 N_\theta^2} = (\overset{r^1 r^2}{\varphi}_{N_\theta^1 N_\theta^2}) \in [H^1_{loc}(I)]^{N_\theta^{1,2}};\ \|\vec{\varphi}_{N_\theta^1 N_\theta^2}\|_{\theta *} < \infty, tr_{h_3^+}(\overset{r^1 r^2}{\varphi}_{N_\theta^1 N_\theta^2}) = 0, r^1 = 0,...,N_\theta^1, r^2 = 0,...,N_\theta^2\}$,
$\vec{H}^\theta_{N_\theta^1 N_\theta^2}(I) = \{\vec{\varphi}_{N_\theta^1 N_\theta^2} = (\overset{r_1 r_2}{\varphi}_{N_\theta^1 N_\theta^2}) \in [L^2(I)]^{N_\theta^{1,2}}; (h_1 h_2)^{-1/2}\overset{r_1 r_2}{\varphi}_{N_\theta^1 N_\theta^2} \in L^2(I), r^1 = 0,...,N_\theta^1, r^2 = 0,...,N_\theta^2\}$, the bilinear forms $R_{\mathbf{N}^{1,2}}$, $a_{\mathbf{N}^{1,2}}$, $b_{N_\theta^{1,2}\mathbf{N}^{1,2}}$, $R^\theta_{N_\theta^{1,2}}$, $b^\theta_{N_\theta^{1,2}}$, $a^\theta_{N_\theta^{1,2}}$ are defined as follows $R_{\mathbf{N}^{1,2}}(\tilde{\vec{v}}_{\mathbf{N}^1\mathbf{N}^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = (\rho\tilde{\mathbf{v}}_{\mathbf{N}^1\mathbf{N}^2}, \mathbf{v}_{\mathbf{N}^1\mathbf{N}^2})_{\mathbf{L}^2(\Omega)}$, $a_{\mathbf{N}^{1,2}}(\bar{\vec{v}}_{\mathbf{N}^1\mathbf{N}^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = a(\bar{\mathbf{v}}_{\mathbf{N}^1\mathbf{N}^2}, \mathbf{v}_{\mathbf{N}^1\mathbf{N}^2})$, $b_{N_\theta^{1,2}\mathbf{N}^{1,2}}(\tilde{\vec{\varphi}}_{N_\theta^1 N_\theta^2}, \vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = b(\tilde{\varphi}_{N_\theta^1 N_\theta^2}, \mathbf{v}_{\mathbf{N}^1\mathbf{N}^2})$, $R^\theta_{N_\theta^{1,2}}(\tilde{\vec{\varphi}}_{N_\theta^1 N_\theta^2}, \vec{\varphi}_{N_\theta^1 N_\theta^2}) = (\chi\tilde{\varphi}_{N_\theta^1 N_\theta^2}, \varphi_{N_\theta^1 N_\theta^2})_{L^2(\Omega)}$, $b^\theta_{N_\theta^{1,2}}(\vec{\varphi}_{N_\theta^1 N_\theta^2}, \tilde{\vec{\varphi}}_{N_\theta^1 N_\theta^2}) = b^\theta(\varphi_{N_\theta^1 N_\theta^2}, \tilde{\varphi}_{N_\theta^1 N_\theta^2})$, $a^\theta_{N_\theta^{1,2}}(\bar{\vec{\varphi}}_{N_\theta^1 N_\theta^2}, \vec{\varphi}_{N_\theta^1 N_\theta^2}) = a^\theta(\bar{\varphi}_{N_\theta^1 N_\theta^2}, \varphi_{N_\theta^1 N_\theta^2})$, for all vector-functions $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}, \bar{\vec{v}}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\tilde{\vec{v}}_{\mathbf{N}^1\mathbf{N}^2} \in \vec{H}_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\vec{\varphi}_{N_\theta^1 N_\theta^2}, \bar{\vec{\varphi}}_{N_\theta^1 N_\theta^2} \in \vec{V}^\theta_{N_\theta^1 N_\theta^2}(I)$, $\tilde{\vec{\varphi}}_{N_\theta^1 N_\theta^2} \in \vec{H}^\theta_{N_\theta^1 N_\theta^2}(I)$, corresponding to $\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}, \bar{\mathbf{v}}_{\mathbf{N}^1\mathbf{N}^2} \in \mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, $\tilde{\mathbf{v}}_{\mathbf{N}^1\mathbf{N}^2} \in \mathbf{H}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$ $\varphi_{N_\theta^1 N_\theta^2}, \bar{\varphi}_{N_\theta^1 N_\theta^2} \in V^\theta_{N_\theta^1 N_\theta^2}(\Omega)$, $\tilde{\varphi}_{N_\theta^1 N_\theta^2} \in H^\theta_{N_\theta^1 N_\theta^2}(\Omega)$, respectively. The linear forms $L_{\mathbf{N}^{1,2}}$ and $L^\theta_{N_\theta^{1,2}}$ are given by the following expressions:

$$\begin{aligned} L_{\mathbf{N}^{1,2}}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}) = \sum_{i=1}^3 \sum_{r_i^1=0}^{N_i^1} \sum_{r_i^2=0}^{N_i^2} \left(r_i^1 + \frac{1}{2}\right)\left(r_i^2 + \frac{1}{2}\right) &\left[\int_{h_3^-}^{h_3^+} \frac{1}{h_1 h_2} \overset{r_i^1 r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i} \left(\overset{r_i^1 r_i^2}{f_i} + \overset{r_i^2}{g}_{\mathbf{N}^1\mathbf{N}^2 i}\Big|_{\Gamma^{1,+}} \gamma^{1,+} + \overset{r_i^1}{g}_{\mathbf{N}^1\mathbf{N}^2 i}\Big|_{\Gamma^{2,+}} \gamma^{2,+} + \right.\right. \\ &\left.\left. + \overset{r_i^2}{g}_{\mathbf{N}^1\mathbf{N}^2 i}\Big|_{\Gamma^{1,-}} \gamma^{1,-}(-1)^{r_i^1} + \overset{r_i^1}{g}_{\mathbf{N}^1\mathbf{N}^2 i}\Big|_{\Gamma^{2,-}} \gamma^{2,-}(-1)^{r_i^2} \right) dx_3 + \sum_{x_3 = h_3^-, h_1 h_2 > 0} \frac{1}{h_1 h_2} \overset{r_i^1 r_i^2}{g}_{\mathbf{N}^1\mathbf{N}^2 i} tr_{h_3^+}(\overset{r_i^1 r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}) \right], \end{aligned}$$

$$L^{\theta}_{N^{1,2}_{\theta}}(\vec{\varphi}_{N^1_\theta N^2_\theta}) = \sum_{r^1=0}^{N^1_\theta}\sum_{r^2=0}^{N^2_\theta}\left(r^1+\frac{1}{2}\right)\left(r^2+\frac{1}{2}\right)\Bigg[\int_{h_3^-}^{h_3^+}\frac{1}{h_1h_2}\overset{r^1r^2}{\varphi}_{N^1_\theta N^2_\theta}\Bigg(\overset{r^1r^2}{f^{\theta}} - \overset{r^2}{g}{}^{\theta}_{N^1_\theta N^2_\theta}\Big|_{\Gamma^{1,+}}\gamma^{1,+} - \overset{r^1}{g}{}^{\theta}_{N^1_\theta N^2_\theta}\Big|_{\Gamma^{2,+}}\gamma^{2,+} -$$

$$-\overset{r^2}{g}{}^{\theta}_{N^1_\theta N^2_\theta}\Big|_{\Gamma^{1,-}}\gamma^{1,-}(-1)^{r_1} - \overset{r^1}{g}{}^{\theta}_{N^1_\theta N^2_\theta}\Big|_{\Gamma^{2,-}}\gamma^{2,-}(-1)^{r_2}\Bigg)dx_3 - \sum_{x_3=h_3^-,\,h_1h_2>0}\frac{1}{h_1h_2}\overset{r^1r^2}{g}{}^{\theta}_{N^1_\theta N^2_\theta}\overset{r^1r^2}{\varphi}_{N^1_\theta N^2_\theta}\Bigg]$$

where $\gamma^{\alpha,\pm} = \sqrt{1+((h^\pm_\alpha)')^2}$, $\Gamma^{\alpha,\pm} = \{x=(x_1,x_2,x_3)\in\Gamma_1;\, x_\alpha = h^\pm_\alpha(x_3)\}$, $\overset{k^1k^2}{v} = \int_{h_1^-}^{h_1^+}\int_{h_2^-}^{h_2^+} vP_{k^1}(z_1)P_{k^2}(z_2)dx_1dx_2$,

$\overset{k^\alpha}{v_\alpha} = \int_{h_\alpha^-}^{h_\alpha^+} v_\alpha P_{k^\alpha}(z_\alpha)dx_\alpha$, for all functions $v\in L^1(\Omega)$, $v_\alpha \in L^1(\Gamma^{3-\alpha,+})\cup L^1(\Gamma^{3-\alpha,-})$, $k^1,k^2\in\mathbf{N}\cup\{0\}$, $\alpha=1,2$,

and vector-function $\mathbf{g}_{\mathbf{N}^1\mathbf{N}^2} = (g_{\mathbf{N}^1\mathbf{N}^2 i})_{i=1}^3$ and function $g^{\theta}_{N^1_\theta N^2_\theta}$ is defined as follows

$$g_{\mathbf{N}^1\mathbf{N}^2 i}(t) = g_i(t) + \sum_{j=1}^{3} tr_{\Gamma_1}\left(\lambda\sum_{p=1}^{3}e_{pp}(\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2 0})\delta_{ij} + 2\mu e_{ij}(\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2 0}) + \eta\zeta_{N^1_\theta N^2_\theta 0}\delta_{ij} + \eta\tau_1\zeta_{N^1_\theta N^2_\theta 1}\delta_{ij}\right)n_j - g_i(0),$$

$$g^{\theta}_{N^1_\theta N^2_\theta}(t) = g^{\theta}(t) - \sum_{j=1}^{3} tr_{\Gamma_1}\left(\kappa\frac{\partial\zeta_{N^1_\theta N^2_\theta 0}}{\partial x_j} + \beta\zeta_{N^1_\theta N^2_\theta 1}\right)n_j - g^{\theta}(0)\,,$$

where $i=1,2,3,$ $\mathbf{n}=(n_i)_{i=1}^3$ is the unit outward normal vector to Γ_1, and $\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2 0}\in\mathbf{V}^2_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, $\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2 1}\in\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, $\zeta_{N^1_\theta N^2_\theta 0}\in V^{\theta,2}_{N^1_\theta N^2_\theta}(\Omega)$, $\zeta_{N^1_\theta N^2_\theta 1}\in V^{\theta}_{N^1_\theta N^2_\theta}(\Omega)$ are restored from the initial conditions $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 0}\in\vec{V}^2_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 1}\in\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\vec{\zeta}_{N^1_\theta N^2_\theta 0}\in\vec{V}^{\theta,2}_{N^1_\theta N^2_\theta}(I)$, $\vec{\zeta}_{N^1_\theta N^2_\theta 1}\in\vec{V}^{\theta}_{N^1_\theta N^2_\theta}(I)$.

For the constructed one-dimensional initial-boundary value problems (10)-(12) the following existence and uniqueness theorem is proved.

Theorem 2. Suppose that functions $h_1^\pm$ and $h_2^\pm$ are such that $\Omega\subset\mathbf{R}^3$ is a Lipschitz domain, $\mu>0$, $3\lambda+2\mu>0,$ $\rho>0$, $\kappa>0$, $\chi>0$, $\tau_0>0$, $\tau_1>0$. If the given functions satisfy the following conditions

$$(h_1h_2)^{-1/2}\overset{r_i^1r_i^2}{f_i}\in L^2(0,T;L^2(I)),\ (h_1h_2)^{-1/2}(\overset{r_i^1r_i^2}{f_i})'\in L^2(0,T;L^2(I)),\ 0\le r_i^\alpha\le N_i^\alpha,\ i=1,2,3,\ \alpha=1,2,$$

$$(\gamma^{\alpha,+})^{3/4}h_{3-\alpha}^{-1/4}\frac{d^k}{dt^k}\overset{r_i^{3-\alpha}}{g}_{\mathbf{N}^1\mathbf{N}^2 i}\Big|_{\Gamma^{\alpha,+}},\ (\gamma^{\alpha,-})^{3/4}h_{3-\alpha}^{-1/4}\frac{d^k}{dt^k}\overset{r_i^{3-\alpha}}{g}_{\mathbf{N}^1\mathbf{N}^2 i}\Big|_{\Gamma^{\alpha,-}}\in L^2(0,T;L^{4/3}(I)),\ k=0,1,2,$$

$$(h_1h_2)^{-1/2}\overset{r^1r^2}{f^{\theta}}\in L^2(0,T;L^2(I)),\ (h_1h_2)^{-1/2}(\overset{r^1r^2}{f^{\theta}})'\in L^2(0,T;L^2(I)),\ 0\le r^\alpha\le N^\alpha,\ \alpha=1,2,$$

$$(\gamma^{\alpha,+})^{3/4}h_{3-\alpha}^{-1/4}\frac{d^k}{dt^k}\overset{r^{3-\alpha}}{g}{}^{\theta}_{N^1_\theta N^2_\theta}\Big|_{\Gamma^{\alpha,+}},\ (\gamma^{\alpha,-})^{3/4}h_{3-\alpha}^{-1/4}\frac{d^k}{dt^k}\overset{r^{3-\alpha}}{g}{}^{\theta}_{N^1_\theta N^2_\theta}\Big|_{\Gamma^{\alpha,-}}\in L^2(0,T;L^{4/3}(I)),\ k=0,1,2,$$

where $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 0}\in\vec{V}^2_{\mathbf{N}^1\mathbf{N}^2}(I),\ \vec{w}_{\mathbf{N}^1\mathbf{N}^2 1}\in\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I),\ \vec{\zeta}_{N^1_\theta N^2_\theta 0}\in\vec{V}^{\theta,2}_{N^1_\theta N^2_\theta}(I),\ \vec{\zeta}_{N^1_\theta N^2_\theta 1}\in\vec{V}^{\theta}_{N^1_\theta N^2_\theta}(I),$ then the initial-boundary value problem (10)-(12) possesses a unique solution.

So, we have constructed an algorithm of approximation of three-dimensional model for thermoelastic bars by well-posed one-dimensional problems and in the following theorem we give the results on the convergence of the algorithm, where we use the following anisotropic weighted Sobolev spaces

$$H^{s,0,0}_{h^{\pm}_{1,2}}(\Omega)=\{v;h_1^k\partial_1^k v\in L^2(\Omega),0\le k\le s\},\ \ H^{0,s,0}_{h^{\pm}_{1,2}}(\Omega)=\{v;h_2^k\partial_2^k v\in L^2(\Omega),0\le k\le s\},\ \ s\in\mathbf{N},$$

$$H^{s,s,1}_{h^{\pm}_{1,2}}(\Omega)=\{v;h_\alpha^{k-1}\partial_\alpha^{k-1}\partial_i^r v\in L^2(\Omega),\alpha=1,2,i=1,2,3,r=0,1,h_\alpha^{k-1}(h_\alpha^{\pm})'\partial_\alpha^k v\in L^2(\Omega),1\le k\le s\},$$

$$\tilde{H}^{s,s,2}_{h^{\pm}_{1,2}}(\Omega)=\{v;h_\alpha^{k-1}\partial_\alpha^{k-1}\partial_i^r\partial_j^{\tilde r} v\in L^2(\Omega),\ i,j=1,2,3,\ r,\tilde r=0,1,h_\alpha^{k-2}\partial_\alpha^k v,h_\alpha^{k-2}(h_\alpha^{\pm})'\partial_\alpha^k v\in L^2(\Omega),$$
$$h_\alpha^{k-1}(h_\alpha^{\pm})''\partial_\alpha^k v\in L^2(\Omega),\ h_\alpha^{k-1}(h_{3-\alpha}^{\pm})'\partial_\alpha^{k-1}\partial_1\partial_2 v,\ h_\alpha^{k-1}(h_\alpha^{\pm})'\partial_\alpha^k\partial_3 v,\ h_\alpha^{k-2}(h_\alpha^{\pm})'(h_\alpha^{\pm})'\partial_\alpha^k v\in L^2(\Omega),$$
$$h_\alpha^{k-1}(h_1^{\pm})'(h_2^{\pm})'\partial_\alpha^{k-1}\partial_1\partial_2 v\in L^2(\Omega),\ 1\le k\le s,\ h_1^{r_1-1}h_2^{r_2-1}\partial_1^{r_1}\partial_2^{r_2}v,\ h_1^{r_1-1}h_2^{r_2-1}(h_\alpha^{\pm})'\partial_1^{r_1}\partial_2^{r_2}v\in L^2(\Omega),$$
$$h_1^{r_1-1}h_2^{r_2-1}(h_1^{\pm})'(h_2^{\pm})'\partial_1^{r_1}\partial_2^{r_2}v\in L^2(\Omega),\ \alpha=1,2,\ r_1,r_2\ge 1,\ r_1+r_2\le s\},\ \ s\ge 2,$$

which are Hilbert spaces equipped with corresponding norms.

Theorem 3. If $\Omega\subset\mathbf{R}^3$ is a bounded domain with Lipschitz boundary, $\mu>0$, $3\lambda+2\mu>0$, $\rho>0$, $\kappa>0$, $\chi>0$, $\tau_0>0$, $\tau_1>0$, $\mathbf{f},\mathbf{f}'\in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g},\mathbf{g}',\mathbf{g}''\in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, $f^\theta, f^{\theta'}\in L^2(0,T;L^2(\Omega))$, $g^\theta, g^{\theta'}, g^{\theta''}\in L^2(0,T;L^{4/3}(\Gamma_1))$, $\mathbf{u}_0\in\mathbf{H}^2(\Omega)\cap\mathbf{V}(\Omega)$, $\mathbf{u}_1\in\mathbf{V}(\Omega)$, $\theta_0\in H^2(\Omega)\cap V^\theta(\Omega)$, $\theta_1\in V^\theta(\Omega)$ satisfy the compatibility conditions (9) and the functions $\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2 0}\in\mathbf{V}^2_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, $\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2 1}\in\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, $\zeta_{N^1_\theta N^2_\theta 0}\in V^{\theta,2}_{N^1_\theta N^2_\theta}(\Omega)$, $\zeta_{N^1_\theta N^2_\theta 1}\in V^{\theta}_{N^1_\theta N^2_\theta}(\Omega)$, corresponding to the initial conditions $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 0}\in\vec{V}^2_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 1}\in\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\vec{\zeta}_{N^1_\theta N^2_\theta 0}\in\vec{V}^{\theta,2}_{N^1_\theta N^2_\theta}(I)$, $\vec{\zeta}_{N^1_\theta N^2_\theta 1}\in\vec{V}^{\theta}_{N^1_\theta N^2_\theta}(I)$ of the one-dimensional problems, tend to $\mathbf{u}_0$, $\mathbf{u}_1$, θ_0 and θ_1 in the spaces $\mathbf{H}^2(\Omega)$, $\mathbf{L}^2(\Omega)$, $H^2(\Omega)$ and $H^1(\Omega)$, respectively, as $N_{\min}=\min_{1\le i\le 3}\{N_i^1,N_i^2,N_\theta^1,N_\theta^2\}\to\infty$, then the one-dimensional problem (10)-(12) possesses a unique solution and the sequences of vector-functions $\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2}(t)$ and functions $\zeta_{N^1_\theta N^2_\theta}(t)$, restored from the solutions $\vec{w}_{\mathbf{N}^1\mathbf{N}^2}(t)$ and $\vec{\zeta}_{N^1_\theta N^2_\theta}(t)$ of the problem (10)-(12), converge to the solution of the three-dimensional problem (6)-(8)

$$\begin{aligned}&\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2}(t)\to\mathbf{u}(t)\ \text{ in }\mathbf{H}^1(\Omega), &&\zeta_{N^1_\theta N^2_\theta}(t)\to\theta(t)\ \text{ in }H^1(\Omega),\\ &\mathbf{w}'_{\mathbf{N}^1\mathbf{N}^2}(t)\to\mathbf{u}'(t)\ \text{ in }\mathbf{L}^2(\Omega), &&\zeta'_{N^1_\theta N^2_\theta}(t)\to\theta'(t)\ \text{ in }L^2(\Omega),\end{aligned}\qquad \text{for all } t\in[0,T],\ \text{as } N_{\min}\to\infty.$$

Furthermore, if $d^r\mathbf{u}/dt^r\in L^2(0,T;(H^{s_r,s_r,1}_{h^{\pm}_{1,2}}(\Omega))^3)$, $\mathbf{u}''\in L^2(0,T;(H^{s_2,0,0}_{h^{\pm}_{1,2}}(\Omega))^3)$ or $\mathbf{u}''\in L^2(0,T;(H^{0,s_2,0}_{h^{\pm}_{1,2}}(\Omega))^3)$, $d^r\theta/dt^r\in L^2(0,T;\ H^{s_r^\theta,s_r^\theta,1}_{h^{\pm}_{1,2}}(\Omega))$, $r=0,1$, $\theta''\in L^2(0,T;H^{s_2^\theta,0,0}_{h^{\pm}_{1,2}}(\Omega))$ or $\theta''\in L^2(0,T;H^{0,s_2^\theta,0}_{h^{\pm}_{1,2}}(\Omega))$, $s_k,s_k^\theta\in\mathbf{N}$, $k=0,1,2$, $s_0,s_0^\theta,s_1,s_1^\theta\ge 2$, and $\mathbf{u}_0\in(\tilde{H}^{\tilde s_0,\tilde s_0,2}_{h^{\pm}_{1,2}}(\Omega))^3$, $\mathbf{u}_1\in(H^{2,2,1}_{h^{\pm}_{1,2}}(\Omega))^3$, $\theta_0\in\tilde{H}^{\tilde s_0^\theta,\tilde s_0^\theta,2}_{h^{\pm}_{1,2}}(\Omega)$, $\theta_1\in H^{\tilde s_1^\theta,\tilde s_1^\theta,1}_{h^{\pm}_{1,2}}(\Omega)$, $\tilde s_0,\tilde s_0^\theta,\tilde s_1^\theta\in\mathbf{N}$, $\tilde s_0,\tilde s_0^\theta\ge 3$, $\tilde s_1^\theta\ge 2$, then, for suitable initial data $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 0}$, $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 1}$, $\vec{\zeta}_{N^1_\theta N^2_\theta 0}$, $\vec{\zeta}_{N^1_\theta N^2_\theta 1}$, the following estimate is valid

$$\begin{aligned}&\left\|\mathbf{u}'-\mathbf{w}'_{\mathbf{N}^1\mathbf{N}^2}\right\|_{C^0([0,T];\mathbf{L}^2(\Omega))}+\left\|\mathbf{u}-\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2}\right\|_{C^0([0,T];\mathbf{H}^1(\Omega))}+\\ &+\left\|\theta'-\zeta'_{N^1_\theta N^2_\theta}\right\|_{C^0([0,T];L^2(\Omega))}+\left\|\theta-\zeta_{N^1_\theta N^2_\theta}\right\|_{C^0([0,T];H^1(\Omega))}\le\frac{1}{(N_{\min})^s}o(T,\Omega,\Gamma_0,h_1^{\pm},h_2^{\pm},\mathbf{N}^1,\mathbf{N}^2,N_\theta^1,N_\theta^2),\end{aligned}$$

where $s=\min\{s_0-3/2,s_1-3/2,s_2,s_0^\theta-3/2,s_1^\theta-3/2,s_2^\theta,\tilde s_0-5/2,\tilde s_0^\theta-5/2,\tilde s_1^\theta-3/2\}$ and $o(T,\Omega,\Gamma_0,h_1^{\pm},h_2^{\pm},\mathbf{N}^1,\mathbf{N}^2,N_\theta^1,N_\theta^2)\to 0$, as $N_{\min}\to\infty$.

მათემატიკა

თერმოდრეკადი ძელების არაკლასიკური მოდელის ერთგანზომილებიანი ამოცანებით აპროქსიმაციის შესახებ

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ნაშრომში განხილულია ცვალებადი კვეთის თერმოდრეკადი ძელების გრინ-ლინდსეის არაკლასიკური მოდელის შესაბამისი სამგანზომილებიანი საწყის-სასაზღვრო ამოცანის ვარიაციული ფორმულირება. ძელის არაკლასიკური დინამიკური სამგანზომილებიანი მოდელისათვის აგებულია ერთგანზომილებიანი ამოცანების მიმდევრობით აპროქსიმაციის ალგორითმი, როცა საზღვრის გვერდით ზედაპირებზე მოცემულია ზედაპირული ძალების სიმკვრივე და სითბოს ნაკადი საზღვრის გარე ნორმალის გასწვრივ. აგებული ერთგანზომილებიანი საწყის-სასაზღვრო ამოცანები გამოკვლეულია სათანადო ფუნქციონალურ სივრცეებში, დამტკიცებულია ერთგანზომილებიანი ამოცანების ამონახსნებიდან აღდგენილი სამი სივრცითი ცვლადის ვექტორ-ფუნქციების მიმდევრობის შესაბამის სივრცეებში კრებადობა საწყისი სამგანზომილებიანი ამოცანის ამონახსნისაკენ და დამატებით პირობებში შეფასებულია აპროქსიმაციის ცდომილება.

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