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Informatics

Main Facets of the Linear Ordering Polytope

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For NP-hard problems, an important direction is the construction and use of facet (facet – face of maximum dimension) inequalities in solving the problem. Note that in this case, with the correct use of all faces, one can obtain a polynomial algorithm for solving the problem. The article presents an overview of the constructed main classes of facets of the polytope of an NP-hard linear ordering problem. The rest of the facet classes are obtained from the main facets classes using mappings. *© 2022 Bull. Georg. Natl. Acad. Sci.*

NP hard problem, linear ordering problem, polytope, facets

Let $G=(N_n,E)$ be a complete orientation graph, where to each edge $(i,j)\in E$ corresponds a weight c_{ij}. Then, in the terminology of graphs, the linear ordering problem can be formulated as follows: find an acyclic tournament of maximum weight in a complete edge-weighted digraph. If each acyclic tournament $T=(N_n,E_T)$ is associated with a point in n^2-n dimensional space as follows:

$$x_{ij}=\begin{cases}1,(i,j)\in E_T\\0,(i,j)\notin E_T\end{cases}$$

then of the linear ordering problem, as an integer linear programming problem, has the form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}x_{ij}\to\max$$

$$0\le x_{ij}\le 1, x_{ij}+x_{ji}=1, i\ne j, i,j=1,\dots,n; \quad (1)$$

$$0\le x_{ij}+x_{jk}-x_{ik}\le 1, i\ne j, i\ne k, j\ne k, i,j,k=1,\dots,n; \quad (2)$$

$$x_{ij}\in[0,1], i\ne j, i,j=1,\dots,n.$$

The polytope corresponding to the system (1), (2) will be denoted by B_n and will be called the initial relaxation polytope. The linear convex hull of the integer vertices of the polytope B_n will be called the linear ordering polytope and will be denoted by P_n. Taking into account the system of equalities $x_{ij} + x_{ji} = 1, i, j = 1,...,n$ polytope B_n and P_n are considered in $\frac{n^2 - n}{2}$ dimensional space.

For the description of the linear ordering polytope by using linear inequalities, a number of important results are obtained in [1]; Mobius ladder inequalities and m-fence inequalities in [2]; m-fence inequalities in [3]; $(m;\ k)$-fence inequalities in [4]; t-reinforced m-fence inequalities in [5]; t-reinforced $(m;\ k)$-fence inequalities, $k \geq 3$ k – odd, $t = 2$, $\tau \geq 4$ in [6]; t-reinforced $(m;\ k)$-fence inequalities, $k \geq 3$ k – odd, $t = 2$, $\tau = 3$ in [7]; the facets using a graph in [8]; new $(m;\ k)$-fence inequalities. Also, in [9] and [10], obtained two types to mapping facets, which plays an important role for obtaining facets of the linear ordering polytope.

Facets induced by Mobius ladders

Definition 1. Let $D = (V, M)$ be a sub digraph of D_n which is generated by k dicycles $C_1, \ldots, C_k$ i. e. $V = \bigcup V(C_i),\ M = \bigcup C_i$, satisfies the following properties:

(a1) $k \geq 3$ and k is odd.

(a2) The length of C_i is three or four, $i = 1, \ldots, k$.

(a3) The degree of each node $u \in V(M)$ is at least three.

(a4) If two dicycles C_i and $C_j, 2 \leq i+1 \prec j \leq k$ have a node, say v, in common then C_j is either left-adjacent or right-adjacent to C_i bat not both.

(a5) Given any dicycle C_j, $j \in \{1, \ldots, k\}$ set $J = \{1, \ldots, k\} \cap (\{j-1, j-3, ..., j-k+2\}) \cup \{j+1, j+3, ..., k-2\})$. Then the set $M - \{e_i, i \in J\}$ contains exactly one dicycle, namely C_j. Then D is called a Mobius ladder.

On the Fig. 1 depicted a Mobius ladder.

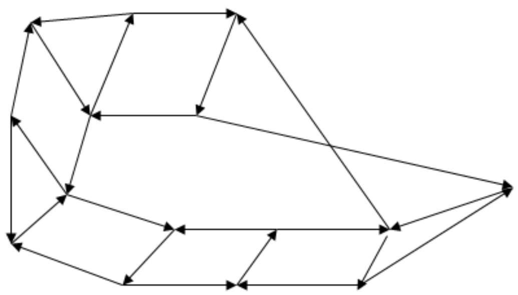

Fig 1. Mobius ladder.

In [1] the following result is obtained:

Theorem 1. Let $D = (V, M)$ be a Mobius ladder in the complete digraph D_n generated by the k dicycle $C_1, \ldots, C_k$. Then the Mobius ladder inequality

$$\sum_{(i,j) \in M} x_{ij} \leq |M| - \frac{k+1}{2} \qquad (3)$$

defines a facet of P_n for $n \geq |V|$.

In the future, in the paper we will draw a directed graph corresponding to non-integer vertices of the polytope B_n, where the value along the edge direction is 1, along the reverse direction of the edge is 0, and the edges that will be absent in the figure are equal to 1/2.

We receive the following results:

Theorem 2. Let $D=(V,M)$ be a Mobius ladder in the complete digraph D_n generated by the k dicycles $C_1,\ldots,C_k$. Then, when deleting all common edges of neighboring orcycles, a graph is obtained that corresponds to a non-integer vertex x^0 of the polytope B_n.

Theorem 3. Let $D=(V,M)$ be a Mobius ladder in the complete digraph D_n generated by the k dicycles $C_1,\ldots,C_k$, and x^0 non- integer vertex of polytope B_n which obtained, if deleting all common edges of neighboring orcycles, when all adjacent integer vertices for x^0 and only they satisfy the inequality (3) as an equality.

m-facets and t-reinforced m-facets

We introduce the notation $x_{ij}+x_{jk}-x_{ik}=(i,j,k),\ i\neq j,\ i\neq k,\ j\neq k,\ i,j,k=1,\ldots,n.$

When describing non-integer vertices of the polytope B_n, the following lemmas are used:

Lemma 1.

$$(i,j,k)=0 \Rightarrow (j,k,i)=0,(k,i,j)=0,(j,i,k)=1,(i,k,j)=1,(k,j,i)=1.$$

Lemma 2.

$$(i,j,k)=0,(i,k,l)=0 \Leftrightarrow (i,j,l)=0,(j,k,l)=0;$$

$$(i,j,k)=1,(i,k,l)=1 \Leftrightarrow (i,j,l)=1,(j,k,l)=1;$$

$$x_{ij}=0,x_{jk}=0 \Leftrightarrow x_{ij}=0,(i,j,k)=0;\quad x_{ij}=1,x_{jk}=1 \Leftrightarrow x_{ij}=1,(i,j,k)=1.$$

Example 1. Consider a Mobius ladder containing three oriented cycles of length four, shown in Fig.2. The facet corresponding to Fig. 2, taking into account the system of equalities $x_{ij}+x_{ji}=1$, $i\neq j, i,j=1,\ldots,6$, can be rewritten like this:

$$x_{14}+x_{25}+x_{36}-x_{15}-x_{26}-x_{34}-x_{16}-x_{24}-x_{35}\leq 1. \tag{4}$$

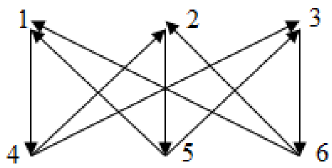

Fig. 2. Mobius ladder containing three oriented cycles of length four.

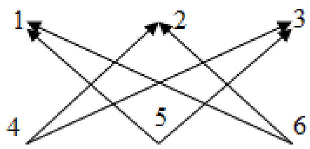

Fig. 3. Non-integer vertex x^0 of the polytope B_n.

Based on Theorem 2, consider a non-integer vertex x^0 of the polytope B_n shown in Fig. 3.

The non-integer vertex x^0 of the polytope B_n shown in Fig. 3 looks like:

$$x^0 = (x_{15} = 0, x_{26} = 0, x_{34} = 0, x_{16} = 0, x_{24} = 0, x_{35} = 0, \ x_{14} = \frac{1}{2}, x_{25} = \frac{1}{2}, x_{36} = \frac{1}{2},$$
$$x_{12} = \frac{1}{2}, x_{23} = \frac{1}{2}, x_{13} = \frac{1}{2}, x_{45} = \frac{1}{2}, x_{56} = \frac{1}{2}, x_{46} = \frac{1}{2}).$$

x^0 satisfies the following system of equalities:

$$x_{15} = 0, x_{26} = 0, x_{34} = 0, x_{16} = 0, x_{24} = 0, x_{35} = 0$$

$$x_{12} + x_{24} - x_{14} = 0, x_{12} + x_{25} - x_{15} = 1 \Leftrightarrow x_{14} + x_{45} - x_{15} = 1, x_{24} + x_{45} - x_{25} = 0 ;$$

$$x_{13} + x_{36} - x_{16} = 1, x_{13} + x_{34} - x_{14} = 0 \Leftrightarrow x_{14} + x_{46} - x_{16} = 1, x_{34} + x_{46} - x_{36} = 0 ;$$

$$x_{23} + x_{36} - x_{26} = 1, x_{23} + x_{35} - x_{25} = 0 \Leftrightarrow x_{25} + x_{56} - x_{26} = 1, x_{35} + x_{56} - x_{36} = 0 .$$

Using Lemma 1 and Lemma 2, the last system of equalities can be rewritten in a convenient form:

$$x_{15} = 0, x_{26} = 0, x_{34} = 0, x_{16} = 0, x_{24} = 0, x_{35} = 0 ,$$

$$(1,5,2) = 0, (1,2,4) = 0 \Leftrightarrow (1,5,4) = 0, (5,2,4) = 0;$$

$$(1,6,3) = 0, (1,3,4) = 0 \Leftrightarrow (1,6,4) = 0, (6,3,4) = 0;$$

$$(2,6,3) = 0, (2,3,5) = 0 \Leftrightarrow (2,6,5) = 0, (6,3,5) = 0 .$$

Adjacent integer vertices for a non-integer vertex x^0 correspond to the following linear orders:

$$\begin{pmatrix}5\\6\end{pmatrix}15\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\6\end{pmatrix}25\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}4\\5\end{pmatrix}36\begin{pmatrix}1\\1\end{pmatrix},$$

$$614253, 625143, \ 514362, \ 536142, \ 425361, \ 436251,$$

where the entry $\begin{pmatrix}i_1\\i_2\end{pmatrix}$ means either $i_1 i_2$ or $i_2 i_1$. It can be checked that, in the polytope B_n, the above given adjacent integer vertices for the non-integer vertex x^0 from Fig. 3 and only they satisfy inequality (4) as an equality.

Definition 2. In a polytope B_n, if only adjacent integer vertices for a non-integer vertex x^0, turn the facet inequality $f(x) \le 0$ of the polytope P_n into equality, then the facet inequality $f(x) \le 0$ will be called exact faceted cutting for non-integer vertex x^0.

Consider a non-integer point corresponding to the graph shown in Fig. 4, where $m \ge 3$.

Theorem 4. Non-integer point

$$x^0 = (x_{i_s i_l} = \frac{1}{2}, x_{j_s j_l} = \frac{1}{2}, s \ne l, s, l = 1, ..., m;$$

$x_{i_s j_s} = \frac{1}{2}, x_{j_s i_s} = \frac{1}{2}, s = 1, ..., m; x_{i_s j_l} = 0, x_{j_l i_s} = 1, s \ne l, s, l = 1, ..., m), m \ge 3$; shown In Fig. 4 is the vertex of the polytope B_n.

In [1] and [2], the following class of faces was constructed independently of each other (Fig. 5):

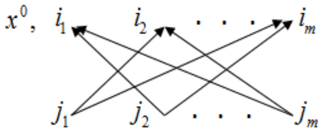

Fig. 4. Each triple of pairs of this graph is the graph shown in Fig. 3.

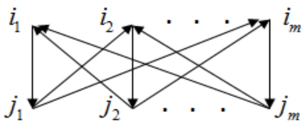

Fig. 5. Each triple of pairs of this graph is the graph shown in Fig. 2.

Theorem 5. Inequality

$$\sum_{s=1}^{m} x_{i_s j_s} - \sum_{s=1}^{m}\sum_{q=1}^{m} x_{i_s j_q} \le 1,$$

where $(i_1, i_2, ..., i_m), (j_1, j_2, ..., j_m)$, $m \le 3$, are disjoint subsets from the set $\{1, 2, ..., n\}$, defines a facet of the polytope P_n.

These facets are called m-fence inequalities or m-facets.

In [4] generalized m-facets and obtained the following facet class:

Theorem 6. Inequality

$$t\sum_{s=1}^{m} x_{i_s j_s} - \sum_{s=1}^{m}\sum_{q=1}^{m} x_{i_s j_q} \le \frac{t(t+1)}{2}, \tag{5}$$

where $(i_1, i_2, ..., i_m), (j_1, j_2, ..., j_m)$ are disjoint subsets from the set $\{1, 2, ..., n\}, 1 \le t \le m-2$, defines a facet of the polytope P_n.

These facets are called t-reinforced m-fence inequalities or t-reinforced m-facets.

Theorem 7. Non-integer vertex x^0 of the polytope B_n, shown in Fig. 4, for $1 \le t \le m-2$, has adjacent integer vertices corresponding only to the following linear orders:

$$J_1 i_{d_1} j_{d_1} i_{d_2} j_{d_2} ... i_{d_\tau} j_{d_\tau} I_1 , \tag{6}$$

where J_1 and I_1 any linear orders from sets $\{j_1, j_2, ..., j_m\} - \{j_{d_1}, j_{d_2}, ..., j_{d_\tau}\}$ and $\{i_1, i_2, ..., i_m\} - \{i_{d_1}, i_{d_2}, ..., i_{d_\tau}\}$;

$$J_2 i_{d_1} j_{d_1} i_{d_2} j_{d_2} ... i_{d_{\tau+1}} j_{d_{\tau+1}} I_2 , \tag{7}$$

where J_2 and I_2 any linear orders from sets $\{j_1, j_2, ..., j_m\} - \{j_{d_1}, j_{d_2}, ..., j_{d_{\tau+1}}\}$ and $\{i_1, i_2, ..., i_m\} - \{i_{d_1}, i_{d_2}, ..., i_{d_{\tau+1}}\}$.

Theorem 8. In the polytope B_n, for any particular value of t, $1 \le t \le m-2$, adjacent integer vertices for the non-integer vertex x^0 depicted in Fig. 4 corresponding to linear orders (6), (7) and only they satisfy inequality (5) as an equality.

Thus, in the polytope B_n, if we have a non-integer vertex x^0 shown in Fig. 4, then we will also have the corresponding, all exact facet cuts (5).

In what follows, in the polytope B_n, to cut off a non-integer vertex x^0, we will use only the corresponding exact facet cuts of the polytope P_n.

(m,k) -facets and t -reinforced (m,k) -facets

As already noted in the previous paragraph, the resulting m-facets are formed using Mobius ladder, which contain 3 orcycles of length 4. And in this section, we will explore the facets formed using Mobius ladder, which contain k orcycles of length 4, where $k \geq 5$, and k is odd.

Definition 3. Let we have numbers $1,2,...,m=\tau k-1$, where addition and subtraction are done mod(m), then the distance from i to j is determined as follows:

$$\rho(i,j)=\rho(j,i)=\min\{j-i,i+\tau k-1-j,i\prec j,i,j=1,...,m\} .$$

Theorem 9. Non-integer vertex

$$x^0=(x_{i_s i_d}=\frac{1}{2},x_{j_s j_d}=\frac{1}{2},s\neq d,s,d=1,...,m;x_{i_s j_s}=\frac{1}{2},x_{j_s i_s}=\frac{1}{2},s=1,...,m;$$

$$x_{i_s j_d}=0,x_{j_d i_s}=1,s\neq d,\rho(s,l)\leq k-1,s,d=1,...,m; \quad (8)$$

$$x_{i_s j_d}=\frac{1}{2},x_{j_d i_s}=\frac{1}{2},\rho(s,d)\geq k,s,d=1,...,m) ,$$

where $m=\tau k-1$, $k\geq 2,\tau\geq 2$; is the vertex of the polytope B_n .

Definition 4. A non-integer vertex (8) of the polytope B_n from Theorem 9 will be called (m,k) - non-integer vertex of the polytope B_n .

Remark 1. The definition of (m,k) -non-integer vertex of the polytope B_n we gave in the case when $m=\tau k-1$, $k\geq 2,\tau\geq 2$, the definition in the general case (m,k) - non-integer vertex of the polytope B_n has the form:

Definition 5. If a point with the coordinates

$$x^0=(x_{i_s i_d}=\frac{1}{2},x_{j_s j_d}=\frac{1}{2},s\neq d,s,d=1,...,m;x_{i_s j_s}=\frac{1}{2},x_{j_s i_s}=\frac{1}{2},s=1,...,m;$$

$$x_{i_s j_d}=0,x_{j_d i_s}=1,s\neq d,\rho(s,l)\leq k-1,s,d=1,...,m;$$

$$x_{i_s j_d}=0\ or 1\ or\ \frac{1}{2},\ x_{j_d i_s}=1\ or\ 0\ or\ \frac{1}{2},\ \rho(s,d)\geq k,s,d=1,...,m) ,$$

where $m=\tau k+q,\ q=-1,...,\tau-2,\ k\geq 2,\tau\geq 2$ is a non-integer vertex of the polytope B_n , then this non-integer vertex will be called (m,k) - non-integer vertex of the polytope B_n .

[3] generalized m-facets and obtained the following class of facets:

Theorem 10. Inequality

$$\sum_{s=1}^{m}x_{i_s j_s}-\sum_{s=1}^{m}\sum_{q=1}^{k-1}(x_{i_s j_{s+q}}+x_{i_s j_{s-q}})\leq\tau-1 , \quad (9)$$

where $(i_1,i_2,...,i_m),(j_1,j_2,...,j_m)$ are disjoint subsets from the set $\{1,2,...,n\}$; $m=\tau k-1$, $k\geq 2,\tau\geq 2$, addition and subtraction of indices is done mod(m); defines a facet of the polytope P_n

Facets (9) are called (m,k) -fence inequalities or (m,k) -facets.

Theorem 11. (m,k) non-integer vertex x^0 of the polytope B_n, from Theorem 9, has adjacent integer vertices corresponding only to the following linear orders

$$J_1 i_{d_1} j_{d_1} i_{d_2} j_{d_2} \dots i_{d_{\tau-1}} j_{d_{\tau-1}} I_1 \ , \tag{10}$$

where J_1 and I_1 any linear orders from sets $\{j_1, j_2, ..., j_m\} - \{j_{d_1}, j_{d_2}, ..., j_{d_{\tau-1}}\}$ and $\{i_1, i_2, ..., i_m\} - \{i_{d_1}, i_{d_2}, ..., i_{d_{\tau-1}}\}$, $\rho(d_s, d_{s+1}) \geq k$, $s = 1, 2, \dots, \tau-1$;

$$J_2 i_{d_1} j_{d_1} i_{d_2} j_{d_2} \dots i_{d_p} j_{d_p} I_2 \ , \tag{11}$$

where J_2 and I_2 any linear orders from sets $\{j_1, j_2, ..., j_m\} - \{j_{d_1}, j_{d_2}, ..., j_{d_p}\}$ and $\{i_1, i_2, ..., i_m\} - \{i_{d_1}, i_{d_2}, ..., i_{d_p}\}$; $p > \tau - 1$, conditions $\rho(d_s, d_{s+1}) \leq k-1$, $s \in \{1, 2, ..., p\}$ are met at one or several places, so that the vertex corresponding to the linear order (11) satisfies inequality (9) as an equality.

Theorem 12. In the polytope P_n, integer vertices corresponding to linear orders (10), (11) and only they, for a specific value of τ, satisfy inequality (9) as an equality.

If we consider the (m,k) facet from [3], where m and k are interconnected as follows: $m = \tau k - 1$, $k \geq 2, \tau \geq 2$; then it is obvious that in [3], the (m,k) facet was constructed only for some values of m. Therefore, the question arises: what form does the (m,k) facet have for other values of m? All the (m,k) facets are constructed, but the main part is not published yet. For example, the class of facets for $m = \tau k$, $k \geq 2, \tau \geq 2$ from [8] is provided:

Theorem 13. Inequality

$$\sum_{s=1}^{m} x_{i_s j_s} - \sum_{s=1}^{m} \sum_{q=1}^{k-1} (x_{i_s j_{s+q}} + x_{i_{s+q} j_s}) + \sum_{d=0}^{k-1} (x_{i_{p+d} j_{p+d+k}} + x_{i_{p+d+k} j_{p+d}}) \leq \tau - 1,$$

where $p = 1, \dots, m$; $\{i_1, i_2, ..., i_m\}$ and $\{j_1, j_2, ..., j_m\}$ are disjoint subsets from the set $\{1, 2, ..., n\}$; $m = \tau k$, $k \geq 2, \tau \geq 2$; addition and subtraction of indices is done mod(m); defines a facet of the polytope P_n.

As already noted, [4] generalized k-facets and obtained t-reinforced m-facets. Similarly, in [5,6,7,8] generalized (m,k)-facets and obtained t-reinforced (m,k)-facets.

For a generalized (m,k) facet, below certain values of τ *and* k, we obtain facets that are fundamentally different from each other. Therefore, taking into account the difference and complexity of individual classes of facets, they are studied separately. For example, the following facets are built separately

For $m = \tau k - 1$, $k \geq 3$, k – odd, $t = 2, \tau = 3$ *;* the facets are published in *[5];*

For $m = \tau k - 1$, $k \geq 3$, k – odd, $t = 2, \tau \geq 4$ *;* the facets are published in *[6];*

For $m = \tau k - 1$, $k \geq 4$, k – even, t – odd, $3 \leq t \leq k-1, \tau = 3$*;*

For $m = \tau k - 1$, $k \geq 5$, k – odd, t – even, $4 \leq t \leq k-1, \tau = 3$*;*

For $m = \tau k - 1$, $k \geq 5$, k – odd, t – even, $4 \leq t \leq k-1, \tau \geq 4$*;*

For $m = \tau k - 1$, $k \geq 4$, k – even, t – odd, $3 \leq t \leq k-1, \tau \geq 4$*;*

For $m = \tau k$, $k \geq 3$, k – odd, $t = 2, \tau = 3$; the facets are published in [8];

For $m = \tau k$, $k \geq 3$, k – odd, $t = 2, \tau \geq 4$;

For $m = \tau k$, $k \geq 4$, k – even, t –odd, $3 \leq t \leq k-1, \tau = 3$;

For $m = \tau k$, $k \ge 5$, k – odd, t –even, $4 \le t \le k-1, \tau = 3$;

For $m = \tau k$, $k \ge 5$, k – odd, t –even, $4 \le t \le k-1, \tau \ge 4$, $\tau \ge 4$;

For $m = \tau k$, $k \ge 4$, k – even, t –odd, $3 \le t \le k-1, \tau \ge 4$.

These classes of facets have been obtained, but the main part of these facets has not been published. All t-reinforced (m,k)-facets are also built for any value of m. These facets have a very complex appearance. However, the principle of constructing these facet classes is similar to each other. Therefore, if we build several classes of t-reinforced (m,k)-facets, then building other t-reinforced (m,k)-facets is already a technical problem.

We have built all the (m,k)-facets and all the t-reinforced (m,k)-facets, but the bulk of these facets have not yet been published.

Thus, having an (m,k) non-integer vertex of the polytope B_n, we also have all the corresponding exact facet cuts.

ინფორმატიკა

ძირითადი ფასეტები წრფივი გადაადგილებების მრავალწახნაგასთვის

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NP–სირთულის ამოცანების მრავალწახნაგებისთვის ფასეტური (ფასეტა–მაქსიმალური გან-ზომილების წახნაგი) უტოლებების აგება და მათი გამოყენება ამოხსნის დროს მნიშვნელოვანი მიმართულებაა. ცხადია, ამ შემთხვევაში შეიძლება ვიფიქროთ ამ ამოცანების ამოხსნის პოლი-ნომიალ ალგორითმებზეც. ნაშრომი შეიცავს NP–სირთულის წრფივი გადაადგილებების მრა-ვალწახნაგას ძირითადი ფასეტების კლასების მოკლე მიმოხილვას. დანარჩენი ფასეტების კლა-სები კი მიიღება ძირითადი ფასეტების კლასებისგან ასახვების საშუალებით.

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