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Mathematics

Consistent and Unbiased Estimators of any Parametric Function for Gaussian Statistical Structures

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In the paper we define consistent and unbiased estimators of any parametric function for Gaussian statistical structures. Necessary and sufficient conditions for the existence of such estimators are given.

orthogonal statistical structure, weakly statistical structure, separable statistical structure, strongly statistical structure

Let measurable space (E,S) and family of $\{\mu_i, i \in I\}$ be given on this space [1]. In the paper we prove necessary and sufficient conditions of consistent and unbiased estimators of any parametric function for Gaussian statistical structures**.**

Definition 1. The set of objects $\{E,S,\mu_i, i \in I\}$ is called a statistical structure.

Definition 2. A statistical structure $\{E,S,\mu_i, i \in I\}$ is called orthogonal (singular) (O) if a family of probability measure $\{\mu_i, i \in I\}$ consists of pairwise singular measures (i.e. $\mu_i \perp \mu_j$, $\forall i \neq j$).

Example 1. Let $E=[0,1]$, S be Borel σ -algebra of [0,1] subsets. Let:

$$\mu_1(B) = 2\ell\left(B \cap \left[0,\frac{1}{2}\right]\right), \quad B \in S,$$

$$\mu_2(B) = 2\ell\left(B \cap \left[\frac{1}{2},1\right]\right), \quad B \in S,$$

$$\mu_3(B) = 3\ell\left(B \cap \left[0,\frac{1}{3}\right]\right), \quad B \in S,$$

where ℓ is Lebesgue measure on S. Then $\mu_1 \perp \mu_2$ and $\mu_2 \perp \mu_3$ but μ_3 is not orthogonal to μ_3.

Definition 3. Statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable (WS), if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that the relations are fulfilled:

$$\mu_i(X_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (i, j \in I).$$

Definition 4. Statistical structure $\{E, S, \mu_i, i \in I\}$ is called separable (S), if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that the relations are fulfilled:

$$1)\ \mu_i(X_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (i, j \in I),$$

$$2)\ \forall i, j \in I:\ card\left(X_i \cap X_j\right) \leq c, \text{ if } i \neq j,$$

where c denotes the continuum power.

Definition 5. Statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly separable (SS), if there exists a disjoint family of S-measurable sets $\{X_i, i \in I\}$ such that the relations are fulfilled:

$\mu_i(X_i) = 1, \quad \forall i \in I$.

Remark 1. From strong separability there follows separability, from separability there follows weak separability and from weak separability it follows orthogonality but not vice versa, i.e.

$$(\mathrm{SS}) \Rightarrow (\mathrm{S}) \Rightarrow (\mathrm{WS}) \Rightarrow (\mathrm{O}).$$

Example 2. Let $E = [0,1] \times [0,1]$, S be Borel σ-algebra of E subsets. Consider the sets

$$X_i = \begin{cases} 0 \leq x \leq 1, & y = i, \quad i \in (0,1]; \\ 0 \leq x \leq 1, & 0 \leq y \leq 1, \quad i = 0 \end{cases}$$

and assume that ℓ_i, $i \in (0,1]$ are linear Lebesgue measures on X_i and L_0 is a plane Lebesgue measure on $[0,1] \times [0,1]$. Then the statistical structure $\{E, S, \ell_i, i \in (0,1]\}$ is orthogonal but not weakly separable.

Let I be the set of parameters and $B(I)$ σ-algebra of subsets of I, which contains all finite subsets of I.

Definition 6. Statistical structure $\{E, S, \mu_i, i \in I\}$ will be said to admit consistent estimators of parameters $i \in I$ (CE), if there exists at least one measurable map $\delta : (E, S) \to (I, B(I))$ such that

$$\mu_i\left(\{x:\ \delta(x) = i\}\right) = 1, \quad \forall i \in I.$$

Definition 7. Statistical structure $\{E, S, \mu_i, i \in I\}$ will be said to admit consistent estimators of any parametric function (PCE) if for any real bounded measurable function $g : (I, B(I)) \to R$ there exists at least one measurable function $f : (E, S) \to R$ such that

$$\mu_i\left(\{x:\ f(x) = g(i)\}\right) = 1, \quad \forall i \in I.$$

Definition 8. Statistical structure $\{E, S, \mu_i, i \in I\}$ will be said to admit unbiased estimators of any parametric function (UPC) if for any real bounded measurable function $g : (I, B(I)) \to R$ there exists at least one measurable function $\beta : (E, S) \to R$ such that

$$\int_E \beta(x)\mu_i(dx) = g(i), \quad \forall i \in I .$$

Remark 2. If statistical structure $\{E,S,\mu_i,i\in I\}$ admitting consistent estimators of parameters $i \in I$, then the statistical structure $\{E,S,\mu_i,i\in I\}$ [2] which admits consistent estimators for any parametric function and statistical structure, which admits unbiased estimators of any parametric function but not vice versa, i.e.:

$$(\text{CE}) \Rightarrow (\text{PCE}) \Rightarrow (\text{UPC}) .$$

Remark 3. If statistical structure $\{E,S,\mu_i,i\in I\}$ admitting consistent estimators of parameters $i \in I$ then this statistical structure $\{E,S,\mu_i,i\in I\}$ is strongly separable but not vice versa, i.e. [3]:

$$(\text{CE}) \Rightarrow (\text{SS})$$

Example 3. Let $E=[0,1]\times[0,1]$, let $B([0,1]\times[0,1])$ be Borel σ -algebra of subsets. As a set of parameters consider the set $I=[0,1]\times[2,3]$. Let us take the $B([0,1]\times[2,3])$ measurable sets

$$X_i = \begin{cases} 0 \le x \le 1, \quad y = i, & \text{if } i \in [0,1], \\ x = i-2, \quad 0 \le y \le 1, & \text{if } i \in [2,3] \end{cases}$$

and denote by μ_i, $i \in [0,1]\cup[2,3]$, linear Lebesgue measure on X_i. Then the statistical structure $\{[0,1]\times[0,1], B([0,1]\times[0,1]), \mu_i, i\in[0,1]\times[2,3]\}$ is a separable statistical structure. Suppose that it admits consistent estimators of parameters.

$$\delta : ([0,1]\times[0,1], B([0,1]\times[0,1])) \to (I, B(I))$$

with

$$\mu_i(\{x:\ \delta(x)=i\}) = 1 ,\ \forall i \in [0,1]\times[2,3] .$$

Let us introduce sets

$$A_1 = \{x:\ \delta(x)\in[0,1]\} \text{ and } A_2 = \{x:\ \delta(x)\in[2,3]\} .$$

It is clear that A_1 and A_2 are $B([0,1]\times[0,1])$-measurable sets and we have:

$$\mu_i(A_1 \cap \{[0,1]\times\{i\}\}) = 1 ,\ \forall i \in [0,1]$$

and

$$\mu_i(A_2 \cap \{\{\tau-2\}\times[0,1]\}) = 1 ,\ \forall i \in [2,3] .$$

Further, according to the Fubin theorem we conclude that $\ell(A_1)=1$ and $\ell(A_2)=1$ (where ℓ is the Lebesgue plane measure). From here, taking into account that

$$A_1 \cap A_2 = \varnothing \text{ and } A_1 \cup A_2 = [0,1]\times[0,1] ,$$

we verity that

$$\ell([0,1]\times[0,1]) = 2 ,$$

which contradicts the fact that $\ell([0,1]\times[0,1])=1$. Hence, thus statistical structure does not admit a consistent estimators of parameters.

Remark 4. The statistical structure $\{[0,1]\times[0,1], \ell_i, i\in[0,1]\}$ in Example 2 is orthogonal, but not weakly separable. Hence this orthogonal statistical structure does not admit a consistent estimator of parameters [4].

The Consistent Estimators of Parameters of Gaussian Statistical Structures

Let $\xi_i(t,\omega)$, $t=(t_1,t_2,\ldots,t_n)\in T$, where T is a closed bounded subset of R^n, $i\in I$, $card\ I=\chi_0$ be a real Gaussian homogeneous field on T with zero mean $E[\xi_i(t)]=0$ $(t\in T,\ i\in I)$ and correlation functions of difference of arguments

$$E[\xi_i(t)\xi_i(s)]=R_i(t-s)\quad (t,s\in T,\ i\in I).$$

Let $\{\mu_i, i\in I\}$ be the corresponding probability measures given on (E,S) and let $f_i(\lambda)$ $\left(\lambda\in R^n\right)$ be bounded densities for all $i\in I$. Let

$$\int\limits_{R^n}\int\limits_{R^n}\frac{\left|\tilde{b}_{i',i''}(\lambda,\mu)\right|^2}{f_{i'}(\lambda)f_{i'}[\mu]}\,d\lambda\,d\mu=+\infty,\quad \forall i'\neq i'',\quad i',i''\in I,$$

where $\tilde{b}_{i',i''}(\lambda,\mu)$ $\left(\lambda,\mu\in R^n,\ \forall i'\neq i'',\ i',i''\in I\right)$ is the Fourier transformation of the following functions

$$b_{i',i''}(t,s)=R_{i''}(t,s)-R_{i'}(t,s)\ \ (\forall i'\neq i'',\ i',i''\in I).$$

Then the corresponding probability measures $\mu_{i'}$ and $\mu_{i''}$ are pairwise orthogonal measures (see [5]) and $\{E,S,\mu_i,i\in I\}$ is Gaussian orthogonal statistical structure.

Theorem 1. Let $\{E,S,\mu_i,i\in I\}$, $card\ I=\chi_0$ be Gaussian orthogonal statistical structure, then

$$(\mathrm{O})\Rightarrow(\mathrm{WS})\Rightarrow(\mathrm{S})\Rightarrow(\mathrm{CE})\Rightarrow(\mathrm{PCE})\Rightarrow(\mathrm{UPC}).$$

Proof. Due to the singularity of statistical structure $\{E,S,\mu_i,i\in I\}$ there exists the family of S-measurable sets $\{X_{ik}\}$ such that for any $i\neq k$: $\mu_k(X_{ik})=0$ and $\mu_i(E-X_{ik})=0$. Therefore, if we consider the sets $X_i=\bigcup\limits_{k\neq i}(E-X_{ik})$, we get $\mu_i(X_i)=0$. Hence, $\mu_i(E-X_i)=1$ On the other hand, for $k\neq i$ we have $\mu_k(E-X_i)=0$. It means that the statistical structure $\{E,S,\mu_i,i\in I\}$ is weakly separable. Therefore, there exists the family of S-measurable sets $\{X_i, i\in I\}$ such that

$$\mu_i(X_j)=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}$$

Consider now the sets

$$\overline{X}_i=X_i-\left(X_i\cap\left(\bigcup_{k\neq i}X_k\right)\right),\ i\in I.$$

It is obvious that these sets are S-measurable disjoint sets and $\mu_i\left(\overline{X}_i\right)=1$, $\forall i\in I$. Let us define the mapping $\delta:(E,S)\to(I,B(I))$ in the following way $\delta\left(\overline{X}_i\right)=i$, $\forall i\in I$. Then we have:

$$\{x:\ \delta(x)=i\}=\overline{X}_i \text{ and } \mu_i\left(\overline{X}_i\right)=\mu_i\left(\{x:\ \delta(x)=i\}\right)=1,\ \forall i\in I.$$

Hence, δ is consistent estimators of parameters.

Let $g:(I,B(I))\to(R,B(R))$ be a real, bounded, measurable function and define the function δ_g as follows $\delta_g(x)=g(\delta(x))$. Then we obtain:

$$\{x:\ \delta_g(x)=g(i)\}=\{x:\ \delta(x)=i\},$$

$$\mu_i(\{x:\ \delta_g(x)=g(i)\})=\mu_i(\{x:\ \delta(x)=i\})=1,\ \ \forall i\in I$$

and

$$\int_E f_g(x)\mu_i(dx)=g(i),\quad \forall i\in I\ .$$

Combining now all of the above, we conclude that:

1) $(\mathrm{O})\Leftrightarrow(\mathrm{WS})\Leftrightarrow(\mathrm{S})\Leftrightarrow(\mathrm{SS})\Leftrightarrow(\mathrm{CE})$;

2) $(\mathrm{O})\Rightarrow(\mathrm{WS})\Rightarrow(\mathrm{S})\Rightarrow(\mathrm{SS})\Rightarrow(\mathrm{CE})\Rightarrow(\mathrm{PCE})\Rightarrow(\mathrm{UPC})$.

Theorem 2. If Gaussian statistical structure $\{E,S,\mu_i,i\in I\}$, $card\ I=\chi_0$ admits consistent estimators of any parametric function, then

$$(\mathrm{PCE})\Rightarrow(\mathrm{CE})\Rightarrow(\mathrm{SS})\Rightarrow(\mathrm{S})\Rightarrow(\mathrm{WS})\Rightarrow(\mathrm{O}).$$

Proof. Since the statistical structure $\{E,S,\mu_i,i\in I\}$, $card\ I=\chi_0$ admits consistent estimators of any parametric function, denote by $f(x)$ one of the corresponding consistent estimators for $I_j(i)$ indicator. Hence, for the sets $\{x: f_i(x)=I_j(i)\}=X_i$, we have:

$$\mu_j(X_i)=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}$$

Therefore the statistical structure $\{E,S,\mu_i,i\in I\}$, $card\ I=\chi_0$ weakly separable, $(\mathrm{WS})\Rightarrow(\mathrm{O})$ and

$$(\mathrm{O})\Rightarrow(\mathrm{WS})\Rightarrow(\mathrm{S})\Rightarrow(\mathrm{SS})\Rightarrow(\mathrm{CE})\Rightarrow(\mathrm{PCE})\Rightarrow(\mathrm{UPC}).$$

Let H be a separable Hilbert space and let $B(H)$ be Borel σ-algebra on it. Let (x,y) denote the scalar product of elements x and y in H. Denote by χ the characteristic function of Gaussian measure μ, i.e.

$$\chi_j(z)=\int_H e^{i(z,x)}\mu_j(dx)=\exp\left\{i(j,z)-\frac{1}{2}(Bz,z)\right\},\quad j\in H,$$

where B is the correlation operator of μ_j. Let us denote by $\{\mu_j,j\in H\}$ the family of all Gaussian measures on $(H,B(H))$ with the same correlation operator B. We consider the Gaussian statistical structure $\{H,B(H),\mu_j,j\in H\}$. Feldman and Gaeck proved that Gaussian measures are either mutually equivalent or orthogonal on $(H,B(H))$. Let us divide the family $\{\mu_j,j\in H\}$ into disjoint families as follows equivalent measures $\mu_{j_1}\sim\mu_{j_2}$ are included in one family, if

$$\sum_{i=1}^{\infty}\frac{(j_1-j_2,\ell_i)^2}{\lambda_i}<\infty,\quad \forall j_1\neq j_2,\ j_1,j_2\in H,$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthogonal basis of the correlation operator B and λ_i are eigenvalues of operator B. In other families the measures μ_{j_1} and μ_{j_2} will be placed, if

$$\sum_{i=1}^{\infty}\frac{(j_1-j_2,\ell_i)^2}{\lambda_i}=+\infty,\quad \forall j_1\neq j_2,\ j_1,j_2\in H.$$

It is clear that Gaussian measures placed in the least families are orthogonal pairs. If we choose now one representative from each family, then we will get Gaussian orthogonal statistical structure $\left\{H, B(H), \mu_j, j \in H\right\}$.

Theorem 3. Let

$$\sum_{i=1}^{\infty} \frac{\left(j_1 - j_2, \ell_i\right)^2}{\lambda_i} = +\infty, \quad \forall j_1 \neq j_2, \;\; j_1, j_2 \in H \;\; and \;\; card\; H = \chi_0,$$

then

$$(\mathrm{O}) \Rightarrow (\mathrm{WS}) \Rightarrow (\mathrm{SS}) \Rightarrow (\mathrm{CE}) \Rightarrow (\mathrm{PCE}) \Rightarrow (\mathrm{UPC}).$$

Theorem 3 is proved as Theorem 1.

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**აკაკი წერეთლის სახელმწიფო უნივერსიტეტი, ზუსტ და საბუნებისმეტყველო მეცნიერებათა ფაკულტეტი, ქუთაისი, საქართველო*
**სოხუმის სახელმწიფო უნივერსიტეტი, საბუნებისმეტყველო მეცნიერებათა, მათემატიკის, ტექნოლოგიებისა და ფარმაციის ფაკულტეტი, თბილისი, საქართველო*
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